UNDERWATER EXPLOSIONS.
TIME INTERVAL BETWEEN SUCCESSIVE EXPLOSIONS

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February 1941

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Introduction.

It has frequently been observed that an underwater explosion is followed after a short time by a second explosion. The time interval may be of the order of a few seconds for large depth charges, whereas for small detonators it is about 20 milliseconds. It is the purpose of this note to submit an explanation of this phenomenon, to calculate the magnitudes of the effects to be expected, and to compare these with available experimental data.

Description of main experimental results.

Little information appears to be available concerning large explosions, but a fair amount of work has been done in connection with small detonators. In these experiments, detonators have been fired at some distance from a 2½" flat-response quartz receiver, and the resulting voltages amplified and spread out on a C.R.O.

Figure 1 shows a typical C.R.O. photograph. Figure 2 shows the same thing using greater amplification. Both photographs show evidence of a number of explosions which become progressively weaker. The time interval between them is not constant but diminisnes slowly with each explosion. The first explosion differs from the rest in that it shows an instantaneous rise in pressure up to its peak value. In all subsequent explosions the pressure increase is more gradual and for the later explosions the pulses have a nearly symmetrical appearance. Between each pair of pulses there is evidence of a small rarefaction which extends over the major part of the interval. This is particularly noticeable in the case of the later explosions shown in Figure 2.

Assumptions made in calculations.

In the following calculations a natural explanation of all these characteristics results from a consideration of what happens to the gaseous products of an explosion as they expand outwards from the instant of the explosion. The following assumptions are made:-

(i) that the explosion takes place in an infinitely short time. Actually, for the type of detonator studied, the time interval is of the order of 2 microseconds. Subsequent consideration shows that there is little change in the size of the bubble in this time interval, so that little error is involved in treating the explosion as instantaneous.

(ii) that the gaseous products do not dissolve in the water to any appreciable extent during the short period concerned.

(iii) that the gaseous products at all times assume the form of a spherical bubble, and behave as a permanent gas.

(iv) that, to a first approximation, the water can be treated as an incompressible fluid. The extent to which this is justified is later considered in the light of the results obtained. The assumption implies that there is no loss of energy by acoustic radiation.

(v) that there is no dissipation of energy by thermal conduction across the face of the bubble.

Suggested explanation of phenomenon.

With these assumptions it is now possible to see in a general way what happens after the explosion. At the instant of explosion a certain amount of gas is instantaneously generated at

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high pressure and temperature. This immediately begins to force outwards the water in contact with it, a motion which is communicated to a lesser extent to all parts of the surrounding fluid. The potential energy initially possessed by the gas bubble by virtue of its pressure is thus gradually communicated to the water in the form of kinetic energy. By reason of the inertia of the water, this motion overshoots the point at which the pressure in the bubble is equal to the external pressure of the liquid. The bubble thereafter becomes rarefied, and its radial motion is slowed up and brought to rest. The external pressure now compresses the rarefied bubble. Again the equilibrium configuration is overshot, and since by hypothesis there has been no loss of energy to the system by radiation or dissipation, it follows that the bubble comes to rest at the same pressure and volume as at the moment of the explosion. The physical aspects of the explosion are therefore reproduced, and theoretically this process goes on repeatedly, with undiminished amplitude. In practice, of course, energy is lost by acoustic radiation and by dissipation, and this causes a progressive diminution of the amplitudes of the successive pulses.

The fact that the C.R.O. records show a series of compressional pulses with only small rarefactions is readily explained. The peak pressure of the bubble is of the order of 4000 atmospheres. The maximum degree of rarefaction of the bubble that can occur is equal to the external pressure of the sea (say 1 atmosphere). Thus although compressions alternate with rarefactions the magnitude of the latter are small in comparison with the former.

The fact that the C.R.O. traces show the successive compressions as isolated peaks, separated by long regions where the amplitude appears to be zero, results from the high degree of asymmetry of the vibrational properties of a bubble when the pressure variations are excessively high. In such cases the vibrations no longer show the sinusoidal character of a bubble vibrating with small amplitude. Immediately after the explosion the high pressure causes a very rapid expansion of the bubble. This, in conjunction with the smallness of the bubble, results in a rapid drop in pressure which soon becomes immeasurably small in comparison with the peak value. The bubble, however, moves relatively slowly when its size is greater, so that the major part of the time interval between two successive compressions is taken up by the bubble in moving at pressures which are insignificant compared with the peak value.

Mathematical development.

The foregoing description of the expanding bubble is now formulated mathematically.

The gaseous products of the explosion form a bubble of radius r_o and pressure p_o, the radius and pressure at any subsequent time t being r(t) and p(t), or simply r, p. The external pressure of the sea is P. It is required in the first instance to determine the variation of p and r with t. This is most easily obtained by writing down the energy equation for the system, which is got as follows:-

Potential Energy.

This is simply the work done by the bubble when it expands from radius r to infinity,

i.e. $$V(r) = \int_r^\infty (p - P)\, 4\pi r^2\, dr \qquad (1)$$

Kinetic Energy.

This is the energy associated with the water by reason of its radial motion. By the conditions of continuity of the medium and spherical symmetry, the radial velocity at a point distant y from the centre of the bubble is $\frac{r^2}{y^2} \cdot \frac{dr}{dt}$

The total K.E. of the medium is therefore

$$K = \int_r^\infty \frac{1}{2} \cdot \rho\, (4\pi y^2\, dy) \cdot \left[\frac{r^2}{y^2} \cdot \frac{dr}{dt}\right]^2 = 2\pi \rho r^3 \dot{r}^2 \qquad (2)$$

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The total energy of the system is therefore

$$V + K = 2\pi\rho r^3 \dot{r}^2 + \int_r^\infty (p - P)\, 4\pi r^2\, dr = A \qquad \text{(a constant)}.$$

The value of A is determined from the fact that at the commencement of the motion when $r = r_o$, $\dot{r} = o$ so that

$$A = \int_{r_o}^\infty (p - P)\, 4\pi r^2\, dr$$

and using this value, the energy equation becomes

$$r^3 \dot{r}^2 = \frac{2}{\rho} \int_{r_o}^{r} [\, p - P\,]\, r^2\, dr \qquad (3)$$

This equation can be integrated when the nature of the dependence of p on r is known. It is assumed that an adiabatic law is obeyed: this of course is implied by our previous assumption that no heat is conducted across the surface of the bubble.

The adiabatic law, $pv^\gamma = \text{const.}$, gives

$$pr^{3\gamma} = p_o r_o^{3\gamma} \qquad (4)$$

Using this relation, equation (3) becomes

$$r^3 \dot{r}^2 = \frac{2}{\rho} \int_{r_o}^{r} \left[p_o \left(\frac{r_o}{r}\right)^{3\gamma} - P \right] r^2\, dr$$

This integrates directly and gives

$$\frac{dr}{dt} = \sqrt{\frac{2\, p_o}{3\,(\gamma - 1)\,\rho}}\ \frac{1}{x^2}\ \sqrt{(1 + \alpha)\, x - \alpha\, x^4 - x^{-3\gamma + 4}} \qquad (5)$$

where

$$x = r/r_o$$

$$\alpha = (\gamma - 1)\, P/p_o$$

This result can now be written

$$t = r_o \sqrt{\frac{3\,(\gamma - 1)}{2}\ \frac{\rho}{p_o}} \int_1^{r/r_o} \frac{x^2\, dx}{\sqrt{(1 + \alpha)\, x - \alpha\, x^4 - x^{-3\gamma + 4}}} \qquad (6)$$

This equation gives the relationship between r and t and specifies the radial motion of the bubble subsequent to the explosion. The pressure inside the bubble at any instant is furnished by equation (6) in conjunction with (4).

Pressure variations inside the bubble.

In order to simplify further calculations, a definite value is now given to γ. For the type of detonators studied experimentally a value $\gamma = 1.3$ is quoted by the makers. The value $\gamma = 4/3$ is particularly convenient mathematically, and is near enough to 1.3, and has therefore been taken.

Equation .....

Equation (6) now becomes

$$t = r_o \sqrt{\frac{\rho}{2 p_o}} \int_1^{r/r_o} \frac{x^2\,dx}{\sqrt{(1+\alpha)\,x - \alpha x^4 - 1}} \quad \text{with } \alpha = P/3\,p_o \qquad (7)$$

The evaluation of this integral is not possible by ordinary methods. Even graphical methods run into difficulties on account of the singularities of the integrand. However, a sufficiently accurate method is available on account of the smallness of α. For the small detonators, α has a value of about 10^{-4}, and over the early part of the region of integration it therefore plays little part. The value of the integrand, in fact, differs from $\frac{x^2}{\sqrt{x-1}}$ by less than 1% up to the point $x = 5$. The same therefore applies to the integral, the integrand being always positive. But when x varies from $x = 1$ to $x = 5$ the internal pressure of the bubble drops from its initial value p_o to $\frac{p_o}{5^4} = \frac{p_o}{625}$. This variation in pressure is probably larger than one is likely to be interested in, the pressures at subsequent points being of little significance. Hence for practical purposes the pressure variations in the early stages correspond to

$$t = r_o \sqrt{\frac{\rho}{2 p_o}} \int_1^{r/r_o} \frac{x^2\,dx}{\sqrt{x-1}}$$

$$= r_o \sqrt{\frac{2\rho}{p_o}} \left\{ \left(\frac{r}{r_o} - 1\right)^{1/2} + \frac{2}{3}\left(\frac{r}{r_o} - 1\right)^{3/2} + \frac{1}{5}\left(\frac{r}{r_o} - 1\right)^{5/2} \right\} \qquad (8)$$

or, expressing this in terms of the internal pressure p,

$$t = r_o \sqrt{\frac{2\rho}{p_o}} \left\{ \left[\left(\frac{p_o}{p}\right)^{1/4} - 1\right]^{1/2} + \frac{2}{3}\left[\left(\frac{p_o}{p}\right)^{1/4} - 1\right]^{3/2} + \frac{1}{5}\left[\left(\frac{p_o}{p}\right)^{1/4} - 1\right]^{5/2} \right\} \qquad (9)$$

Time intervals between pulses.

Equation (9) cannot be used to determine the time interval T between successive pulses. To determine this it is necessary to revert to the general formula (7) and to carry out the integration over a complete cycle. In this formula the denominator of the integrand

$$(1+\alpha)\,x - \alpha x^4 - 1$$

becomes zero for two positive values of x. One of those values is $x = 1$. The other is a large number, which is denoted by x_1. The physical significance of these zeros is that they correspond to the turning points of the vibrations of the bubble. When $x = 1$ the bubble is in its highest state of compression and when $x = x_1$ it is in its highest state of rarefaction. Furthermore, the motion during compression corresponds completely with the motion during expansion, except for sign. Hence, if the integral of equation (7) is evaluated between the limits $x = 1$ and $x = x_1$ the result will be $t = T/2$. Hence

$$T = r_o \sqrt{\frac{2\rho}{p_o}} \int_1^{x_1} \frac{x^2\,dx}{\sqrt{(1+\alpha)\,x - \alpha x^4 - 1}} \qquad (10)$$

where x_1 is the larger positive root of

$$(1+\alpha)\,x - \alpha x^4 - 1 = 0.$$

The approximate evaluation of (10) is again possible by reason of the smallness of α. Details of this evaluation are given in appendix 2. The result is

$$T = r_o \sqrt{\frac{2\rho}{p_o}} \cdot \frac{2I}{3} \cdot \alpha^{-5/6} \qquad (11)$$

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where

$$I = \int_0^{\pi/2} \sin^{2/3}\theta \, d\theta = 1.12$$

substituting for α this becomes

$$T = \frac{2\sqrt{2}\,I}{3^{1/6}} \rho^{1/2} r_o p_o^{1/3} p^{-5/6} \qquad (12)$$

It may be shown that the error involved in the derivation of this relation is of the order $\alpha^{1/3}$, i.e. about 1/20 for the detonators studied.

Equation (12) was derived on the assumption that $\gamma = 4/3$. The corresponding formula when γ has any value may be shown to be

$$T = I \left(\frac{8}{3}\right)^{1/2} (\gamma - 1)^{-1/3} \rho^{1/2} r_o p_o^{1/3} p^{-5/6} \qquad (12a)$$

Equation (12a) can be expressed in alternative and more convenient forms as follows.

The equation of state for the gas in the bubble gives

$$m R \theta_o = p_o v_o = \frac{4\pi}{3} r_o^3 p_o \qquad (13)$$

where m = quantity of gas in gm. molecules

R = universal gas constant = 8.2 10^7 ergs.

θ_o = initial gas temperature on absolute scale.

By virtue of (13) the term $r_o p_o^{1/3}$ can be eliminated from (12a) with the result

$$T = \left\{ \frac{32}{3\pi^2 (\gamma - 1)^2} \right\}^{1/6} I \rho^{1/2} (m R \theta_o)^{1/3} p^{-5/6} \qquad (12b)$$

T can alternatively be expressed in terms of the initial potential energy of the bubble w_o. For w_o is the work done by the bubble in expanding and is

$$w_o = \int_{v_o}^{\infty} p\,dv = p_o v_o^{\gamma} \int \frac{dv}{v^{\gamma}} = \frac{p_o v_o}{\gamma - 1}$$

$$= \frac{4\pi}{3(\gamma - 1)} \cdot r_o^3 p_o \qquad (14)$$

Eliminating $r_o p_o^{1/3}$ between (14) and (12a)

$$T = I \left(\frac{32}{3\pi^2}\right)^{1/6} \rho^{1/2} w_o^{1/3} p^{-5/6} \qquad (12c)$$

From equation (12c) the following conclusions are drawn.

(1) T is proportional to $w_o^{1/3}$. Since, for a given detonator, the successive pulses in practice become weaker, so that the potential energy w_o at the stages of maximum compression becomes less and less, it follows that T should decrease slowly with each successive pulse. This agrees with the facts described earlier.

(2) In the case of the initial pulse w_o may be identified with the total energy of the explosion. The interval between the first and second pulses is therefore proportional to the cube root of the mass value of the charge, irrespective of the nature of this charge.

(3) .....

(3) T varies inversely with the 5/6 power of the total external pressure. Shorter time intervals are therefore to be expected if a charge is fired at any appreciable depth in the sea.

Pressure variations outside the bubble.

Equation (9) shows how the pressure inside the bubble varies with time. This variation is not readily measured. What is frequently measured is the form of the pressure pulse at a distant point outside the bubble. The form of this pulse is not the same as that of the internal pressure, and the nature of the difference is now considered.

At any instant, t, the expanding bubble functions as a source whose strength S is equal to the rate at which water is being forced outwards, i.e.

$$S = \frac{d}{dt}\left(\frac{4\pi}{3} r^3\right) \tag{15}$$

The velocity potential, χ, at a point distant a from the bubble is, by the usual formula,

$$\chi = \frac{S}{4\pi a} = \frac{1}{4\pi a} \frac{d}{dt}\left(\frac{4\pi}{3} r^3\right) = \frac{1}{a} \frac{d}{dt}\left(\frac{r^3}{3}\right) \tag{16}$$

No retarded potential is here involved as the medium is assumed incompressible, and the velocity of sound is therefore infinite. The pressure at the point a is, therefore, by the usual hydrodynamic formula

$$\Pi = \rho \frac{\partial \chi}{\partial t} + \frac{1}{2} \rho v^2$$

where v is the radial velocity at a

$$= \frac{\rho}{a} \frac{d^2}{dt^2}\left(\frac{r^3}{3}\right) + \frac{1}{2} \rho \left[\frac{r^2 \dot{r}}{a^2}\right]^2 \tag{17}$$

The second term on the right hand side varies inversely with a^4, and therefore rapidly becomes negligible in comparison with the first term which varies inversely with a. Equation (17) therefore simplifies in practice to

$$\Pi = \frac{\rho}{a} \frac{d^2}{dt^2}\left(\frac{r^3}{3}\right) \tag{18}$$

This equation, together with the solution for r furnished by equation (8), specifies the values of Π for any time t. This, however, is not a convenient solution, and a more useful result can be obtained in the following way:-

Equation (18) can be written

$$r^2 \ddot{r} + 2 r \dot{r}^2 = \frac{a\Pi}{\rho} \tag{19}$$

Differentiation of the energy equation (3) gives

$$r^2 \ddot{r} + \frac{3}{2} r \dot{r}^2 = \frac{r}{\rho}(p - P) \tag{20}$$

Subtraction of (20) from (19) gives

$$\Pi = \frac{r}{a}\left\{(p - P) + \frac{1}{2}\rho \dot{r}^2\right\} \tag{21}$$

Substituting in (21) the value for $\dot{r}$ given by (5), and taking $\gamma = 4/3$ gives

$$\Pi = \frac{r}{a}\left\{p - P + \frac{p_0}{x^4}\left[(1 + \alpha)\, x - \alpha x^4 - 1\right]\right\} \tag{22}$$

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$$p = \frac{p_0}{x^4} \quad \text{from (4)}$$

$$x = \frac{r}{r_0} \quad \left.\right\} \text{from (5)}$$

$$\text{and} \quad P = 3 p_0 \alpha \quad \left.\right\}$$

Inserting these values in (22)

$$\Pi = p_0 \frac{r_0^3}{ar^2} \{ (1 + \alpha) - 4 \alpha x^3 \} \tag{23}$$

In the early stage of the motion, which is now considered, the terms α and αx^3 can be neglected in comparison with unity, and then

$$\Pi = p_0 \frac{r_0^3}{3r^2} \tag{24}$$

This is a more useful formula than (18).

The peak value Π_0 of the external pressure corresponds to $r = r_0$, and is

$$\Pi_0 = p_0 \left(\frac{r_0}{3}\right) \tag{25}$$

Equation (24) can now be written as

$$\Pi = \Pi_0 \left(\frac{r_0}{r}\right)^2 \tag{26}$$

so that, using equation (8)

$$t = r_0 \sqrt{\frac{2\rho}{p_0}} \left\{ \left\{ \left[\frac{\Pi_0}{\Pi} \right]^{1/2} - 1 \right\}^{1/2} + \frac{2}{3} \left\{ \left[\frac{\Pi_0}{\Pi} \right]^{1/2} - 1 \right\}^{3/2} + \frac{1}{5} \left\{ \left[\frac{\Pi_0}{\Pi} \right]^{1/2} - 1 \right\}^{5/2} \right\} \tag{27}$$

which shows how Π varies with t in the early stages.

The maximum amplitude of the rarefaction at a distant point is deducible from the more accurate equation (23). From formula (10) it appears that the maximum expansion of the bubble corresponds to $x = x_1$, where x_1 is the larger positive root of

$$(1 + \alpha) x - \alpha x^4 - 1 = 0$$

and is approximately $x_1 = \alpha^{-1/3}$ when α is very small. The corresponding value of Π is given by equation (23) as

$$\Pi_{min} = - \frac{3 p_0 r_0^3}{a r^2}$$

this variation being superposed upon the existing pressure P. Substituting in the above $r = x_1 r_0 = r_0 \alpha^{-1/3}$

$$\Pi_{min} = \frac{3 p_0 r_0}{a} \alpha^{2/3}$$

$$= - 3 \Pi_0 \alpha^{2/3} \qquad \text{(from 25).}$$

The maximum amplitude of the rarefaction is therefore a fraction $3\alpha^{2/3}$ of the initial positive pressure pulse.

Comparison .....

Comparison with experiment.

Experiments made with small 1 gm. detonators have enabled the foregoing theoretical results to be tested in various ways.

(a) Shape of pulses.

Detailed calculations of the form of the external pressure pulses have been made, and on the basis of the results the appearance of the first two pulses has been sketched in Figure 3. In these calculations, equation (27) was used in order to obtain the form of the pulses down to a pressure amplitude of $\Pi_0/100$, a separate estimation of the maximum rarefaction being made.

Figure 1 shows a typical C.R.O. record of a detonator explosion. The receiver employed was a small 2½" quartz hydrophone calculated to have a principal resonance at about 400 kc/s. and aimed to have a flat response at lower frequencies. The receiver was used in conjunction with an amplifier having a high input impedance (10 – 20 megohms) and wide band frequency response (10 c/s to 100 kc/s). The deflections on the C.R.O. records are proportional to the voltage variations across the receiver and therefore, for a perfect receiver, proportional to the pressure variations Π in the water.

In a general way the appearance of the C.R.O. trace agrees with that calculated, as regards the sharpness and isolated character of the pulses. There is one difference, however; on the C.R.O. record the initial pressure rise is followed after a very short time by a small sharp negative pulse. This effect is more apparent in Figure 2 where more amplification has been used. Here it appears that the negative peak is only associated with the first few explosions; it is absent from later explosions where the pressure variations appear to accord more nearly with theory. This effect may be due to the failure of the receiver to respond reliably to the very high frequencies which would be associated with the initial explosions.

(b) Amplitude of initial pressure pulse.

Equation (25) enables the peak pressure amplitude Π_0 at any external point to be calculated. In Appendix I the following data are derived for the detonators

$$p_0 = 3700 \text{ atm.} \qquad r_0 = 0.62 \text{ cms.}$$

Substituting these in equation (25), absolute values are got for the pressure amplitudes: it is calculated, for example, that the pressure at a distance of 10 feet rises to 7.5 atmospheres. Experimentally, a value of about 3 atmospheres is obtained at the same range. Accurate agreement is here not to be expected, for the calculated value is very materially dependent upon p_0 and r_0, reliable values for which are not available.

(c) The time interval T between the first and second pulses.

Equation (12b) has been used in order to get a theoretical value for T. Taking, for example, the case when a detonator is fired at a depth of 15 feet below the surface of the sea, the total pressure P at this depth is found to be

$$P = 1.45 \text{ atm.} = 1.45\ 10^6 \text{ dynes/cm}^2$$

In addition, the makers specify for their detonators

$$\theta_0 = 3000^\circ \text{ C} = 3273^\circ \text{ K.}$$

$$m = 0.0136 \text{ corresponding to 300 ccs. of gas (Appendix I).}$$

$$\gamma = 1.30.$$

Using also the known values

$$R = 8.3\ 10^7 \text{ ergs/gm. molecule.}$$

$$J = 1.12.$$

formula .....

formula (12c) now gives

$$T = 18.6 \text{ milliseconds}$$

The corresponding value obtained by direct measurement is

$$T = 17 \text{ milliseconds}$$

By the same method the interval T can be calculated for the case of a depth charge. For this purpose equation (12c) is more convenient. Consider for example the case of a 300 lb. charge of T.N.T. The heat value of T.N.T. is given as 1040 calories/gm., from which the total energy W_o of the entire charge is found to be $5.9\ 10^{15}$ ergs. At a depth of (say) 30 feet $P = 1.9$ atm. = $1.9\ 10^6$ dynes/cm^2. Equation (12c) then gives

$$T = 1.2 \text{ seconds.}$$

An experimental value

$$T = 0.6 \text{ seconds}$$

has been quoted for a 300 lb. mine exploding at a depth of 30 feet.

(d) Variation of T with pressure P.

As already pointed out, the dependence of T upon P expressed by equation (12b) implies that T should vary with the depth at which a charge is fired. Values of T for the detonators have been worked out for various depths up to 300 feet just as in the example worked out above for 15 feet. The resulting values define the continuous curve drawn in Figure 5. To test these values, direct measurements of T were carried out at sea down to a depth of 245 feet. These experimental values are set out in Table 1 and are marked by the circles in Figure 5. These values lie close to the theoretical curve, showing that equation (12b) gives a reliable figure for the absolute value of T under external pressures P ranging from 1 to 8 atmospheres.

Theoretically T should vary inversely with $P^{5/6}$. To test the accuracy with which the index 5/6 is operative the experimental values for T and P have been plotted on a logarithmic scale in Figure 6. A straight line of slope $-5/6$ has been drawn passing through these points. For purpose of comparison lines with slopes $-2/3$ and -1 have also been indicated. It is evident that the experimental values discriminate quite sharply in favour of the index $-5/6$.

(e) Time intervals between successive pulses.

As was mentioned earlier, it is found experimentally that the time intervals between successive pulses get shorter and shorter, and this effect is in qualitative agreement with theory. It is now possible to make quantitative comparison.

Equation (12a) shows that the time interval between the first and second pulses is proportional to $r_o\, \rho_o^{1/3}$ where r_o and ρ_o refer to the initial values of r and ρ at the first explosion. For a given mass of gas, by virtue of equation (4), r_o is proportional to $\rho_o^{-1/4}$ (assuming $\gamma = 4/3$) so that T depends on the initial pressure of the gas according to the relation

$$T \propto \rho_o^{1/12}$$

But from equations (4) and (25)

$$\rho_o \propto \Pi_o^{4/3}$$

ρ_o being the internal and Π_o the external pressure, so that

$$T \propto \Pi_o^{1/9}$$

This is strictly accurate only if the vibrations are undamped, Π_o being then the same for the first and second pulses. For a damped train of pulses it is not quite legitimate to apply this formula to the pressure amplitude for the first pulse; it is more accurate to regard the first half of the

period .....

period as being determined by the pressure Π_1 of the first pulse and the second half as determined by the value Π_2 for the second pulse. The time interval can then be taken to be proportional to $(\Pi_1^{1/9} + \Pi_2^{1/9})$. In practice little difference is involved, otherwise equation (12a) would require correction for damping. The above result can be generalised for any two consecutive pulses of a train arising from a single detonator. If Π_n, Π_{n+1} are the amplitudes of the nth, (n + 1)th pulses and $T_{n, n+1}$ the time interval between them, then

$$T_{n, n+1} \propto (\Pi_n^{1/9} + \Pi_{n+1}^{1/9}) \qquad (31)$$

This relation has been tested against experimental figures. Relative values of Π_n appropriate to successive pulses have been measured off from the C.R.O. records. This cannot be done satisfactorily from one single photograph because Π_n decreases too rapidly with n to allow of accurate measurement over a number of pulses. However, a series of photographs were taken in which the amplification was progressively increased. These enabled mean values to be given to the ratios Π_n/Π_{n+1} from various pairs of consecutive pulses. In this way it was found that the pressure amplitudes of the first five pulses were in the ratio

$$\Pi_1 : \Pi_2 : \Pi_3 : \Pi_4 : \Pi_5$$

$$= \quad 45 : \quad 30 : \quad 6 : \quad 2 : \quad 1$$

From equation (31) the corresponding time intervals are calculated to be in the ratio

$$T_{12} : \quad T_{23} : \quad T_{34} : \quad T_{45}$$

$$= \quad 1.45 : \quad 1.29 : \quad 1.12 : \quad 1.00$$

The corresponding values got by direct measurement from Figure 2 are

$$1.67 : \quad 1.23 : \quad 1.10 : \quad 1.00$$

Agreement is considered satisfactory in view of the fact that complete reliance cannot be placed in the amplitudes of the pulses shown in Figure 2. The initial pulses are richer than the later ones in high frequency sound of the order of several hundred Kc/s., and the receiver and amplifier do not necessarily reproduce them in true proportion to the others. This would affect the calculation of T_{12} more than the other intervals.

Considerations of the assumptions made.

The foregoing results have been derived on the basis of certain assumptions. These assumptions are now considered in the light of the results obtained

(a) The compressibility of the water.

It is legitimate to treat the water as an incompressible fluid only in so far as the radial velocity of the bubble is small compared with that of sound in water. From equation (8) it may be shown that the maximum radial velocity of the bubble occurs when $\frac{r}{r_0} = \frac{4}{3}$, and that this maximum velocity is

$$\frac{3}{16}\sqrt{\frac{6\,p_0}{\rho}}$$

For the small detonators this amounts to $2.7\ 10^4$ cms/second. The velocity of sound is $15\ 10^4$ cms/second, so that the radial velocity is at most 1/6 that of sound. The assumption of incompressibility is therefore quite reasonable as a first approximation when it is remembered that this maximum velocity is operative for only a relatively short part of the expansion.

(b) The finite duration of the explosion.

It was assumed that the explosion took place instantaneously, whereas in fact for the detonators it lasts about 2 micro seconds. Assuming an instantaneous explosion, the expansion which takes place in the first 2μ seconds is now considered. Equation (8), for small values of t, can

be .....

be written

$$\frac{r}{r_o} = 1 + \frac{1}{2} t^2 \frac{p_o}{\rho r_o^2}$$

Inserting actual values it is found that, after 2 μ seconds

$$\frac{r}{r_o} = 1.018.$$

There is thus very little expansion in this time interval, so that it makes little difference whether the explosion occurs at the beginning of this interval or is distributed over it.

(c) The radiation of acoustic energy.

It was earlier assumed that there was no radiation of sonic energy. Taking the solution obtained on this assumption as a first approximation, it is now possible to calculate what energy is actually radiated when account is taken of the finite speed of sound, c.

The instantaneous flux of sonic energy per unit area per unit time at a range a is $\frac{\Pi^2}{\rho c}$. Substituting for Π from equation (24) this becomes $\frac{p_o^2 r_o^6}{\rho c\, a^2 r^4}$. Integrating over the surface of a sphere of radius a, the rate of emission of energy from the bubble becomes

$$\frac{dx}{dt} = \frac{4\pi p_o^2 r_o^6}{\rho c\, r^4} = \frac{4\pi p_o^2 r_o^2}{\rho c} \cdot \frac{1}{x^4}$$

The energy emitted in the course of the first pulse is therefore

$$W = \frac{4\pi p_o^2 r_o^2}{\rho c} \int_{x=1}^{x=x_1} \frac{dt}{x^4} \qquad (28)$$

where x_1 has the same significance as in equation (10).

Putting $\frac{r}{r_o} = x$ in equation (8) and differentiating, gives

$$dt = r_o \sqrt{\frac{\rho}{2 p_o}} \cdot \frac{x^2\, dx}{\sqrt{x-1}}$$

and substituting this in (28) gives

$$W = 2\pi\sqrt{2}\, \frac{r_o^3 p_o^{3/2}}{c \rho^{1/2}} \int_1^{x_1} \frac{dx}{x^2\sqrt{x-1}} \qquad (29)$$

In equation (29) the major part of the integral arises from values of x near to unity. This fact justifies the use of equation (8) over the range of integration $x = 1$ to $x = x_1$ for equation (8) is valid only over the initial part of this interval ($x = 1$ to $x = 5$). The error introduced on this account is extremely small, and for the same reason it is quite legitimate, and more convenient to write the limits of integration in (29) as $x = 1$ to $x = \infty$. Then

$$W = 2\pi\sqrt{2}\, \frac{r_o^3 p_o^{3/2}}{c \rho^{1/2}} \int_1^{\infty} \frac{dx}{x^2\sqrt{x-1}}$$

$$= \pi^2\sqrt{2}\, \frac{r_o^3 p_o^{3/2}}{c \rho^{1/2}} \qquad (30)$$

This is now to be compared with the total energy W_o available at the moment of the explosion. Taking $\gamma = 4/3$, this latter by virtue of equation (14) is $4\pi r_c^3 p_o$, so that the fraction of the

initial .....

initial energy radiated in the first pulse is

$$\frac{W}{W_0} = \frac{\pi}{2\sqrt{2}} \frac{1}{c} \sqrt{\frac{G}{\rho}} \qquad (32)$$

For the small detonators this works out to be 0.43.

The appreciable magnitude of this figure implies a considerable degree of damping of the vibrations. This fact will affect to some extent the accuracy of the expressions obtained for p (t) and Π (t). The deviations arising will be cumulative and will not impair greatly the validity of the formulae in the earlier stages of the expansion. The effect will be to limit the range of validity of these formulae. The effect on the formulae for T will be quite small (compare with this the fact that the period of any simple harmonic vibratory system is little affected by quite heavy damping).

Formula (32) shows that the damping increases with p_0, so that the amplitude of the second pulse should become relatively less. It is possible, therefore, that if p_0 is made large enough the second pulse may be too weak to be noticed in comparison.

Appendix I .....

APPENDIX I.

Detonator Figures.

The following is the information furnished by the makers concerning the detonators used. The amount of explosive used is 1 gm., part of which is lead azide. The volume of gas formed, at N.T.P., is 300 ccs. The initial temperature θ_0 is 3000° C (= 3273° K). The effective value of γ is 1.3.

The value of p_0 is not given, but, once the initial volume v_0 is known, it may be deduced from the equation of state (13). In this equation m is readily determined, for one gm. molecule of any gas at N.T.P. occupies 22 litres, so that the 300 ccs. of gas resulting from the explosion correspond to a value

$$m = \frac{300}{22000} = 0.0136.$$

The value of v_0 is not given, but is estimated to be about 1 cc. Inserting these values in (13) gives

$$p_0 = 3700 \times 10^6 \text{ dynes/cm}^2.$$

The accuracy of this value depends on the accuracy with which v_0 is known.

The total mechanical energy of the detonator explosion is, from equation (14), $\frac{p_0 v_0}{\gamma - 1}$.

By reason of equation (13) this can be written $\frac{m R \theta_0}{\gamma - 1}$. On substituting the values for m, θ_0 and γ already quoted, a figure of 1200 Joules is obtained for the energy of the detonator.

Appendix II .....

APPENDIX II.

Evaluation of the integral for I.

This integral is

$$K(\alpha) = \int_1^{x_1} \frac{x^2\,dx}{\sqrt{(1+\alpha)\,x - \alpha\,x^4 - 1}}$$

where x_1 is the larger positive root of

$$(1+\alpha)\,x - \alpha\,x^4 - 1 = 0$$

The integral defines a function of α. The asymptotic form of this function as $\alpha \to 0$ is now calculated.

When $\alpha \to 0$ the value of x_1 approaches $\alpha^{-1/3}$. A number x_2 is now chosen which is of a lower order than x_1, but still large compared with unity. This can always be done, and such a number would be $x_2 = \alpha^{-q}$ where q is positive and less than 1/3. Then $K(\alpha)$ may be written in the form

$$K(\alpha) = \int_1^{x_2} \frac{x^2\,dx}{\sqrt{(1+\alpha)\,x - \alpha\,x^4 - 1}} + \int_{x_2}^{x_1} \frac{x^2\,dx}{\sqrt{(1+\alpha)\,x - \alpha\,x^4 - 1}} \qquad (33)$$

This is now compared with the function

$$K_1(\alpha) = \int_1^{x_2} \frac{x^2\,dx}{\sqrt{x-1}} + \int_{x_2}^{\alpha^{-1/3}} \frac{x^2\,dx}{\sqrt{x - \alpha\,x^4}} \qquad (34)$$

It can be shown that, by virtue of the choice of x_2, the relative differences between corresponding terms of (33) and (34) get less and less as $\alpha \to 0$. Hence the limiting forms of both $K(\alpha)$ and $K_1(\alpha)$ are the same.

The integration of the expression for $K_1(\alpha)$ can be carried out. Putting $\sin^2\theta = \alpha\,x^3$ in the second integral, the result is

$$K_1(\alpha) = 2\left\{(x_2-1)^{1/2} + \tfrac{2}{3}(x_2-1)^{3/2} + \tfrac{1}{5}(x_2-1)^{5/2}\right\} + \tfrac{2}{3}\alpha^{-5/6}\int_{\sin^{-1}(\alpha x_2^3)^{1/2}}^{\pi/2} \sin^{2/3}\theta\,d\theta.$$

But since $x_2 \gg 1$ it is possible to write x_2 for $(x_2 - 1)$ in the first term. Then

$$K_1(\alpha) = 2\left\{x_2^{1/2} + \tfrac{2}{3}x_2^{3/2} + \tfrac{1}{5}x_2^{5/2}\right\} + \tfrac{2}{3}\alpha^{-5/6}\left\{\int_0^{\pi/2} \sin^{2/3}\theta\,d\theta - \int_0^{\sin^{-1}(\alpha x_2^3)^{1/2}} \sin^{2/3}\theta\,d\theta\right\} \qquad (35)$$

But since $x_2 \ll \alpha^{-1/3}$, $(\alpha x_2^3)^{1/2} \ll 1$ so that the last integral on the right hand side simplifies to

$$\int_0^{(\alpha x_2^3)^{1/2}} \theta^{2/3}\,d\theta = \tfrac{3}{5}(\alpha x_2^3)^{5/6}$$

Putting .....

Putting this result in (35)

$$K_1(\alpha) = 2\left\{x_2^{1/2} + \frac{2}{3}x_2^{3/2}\right\} + \frac{2I}{3}\alpha^{-5/6} \qquad (36)$$

where

$$I = \int_0^{\pi/2} \sin^{2/3}\theta \, d\theta \doteq 1.12$$

But if x_2 is assumed to have the value α^{-q} then

$$\frac{x_2^{3/2}}{\alpha^{-5/6}} = \alpha^{5/6-3q/2}$$

and since $q < 1/3$ this is a positive power of α, whose value accordingly tends to zero as $\alpha \to 0$. Thus the first term of (36) finally becomes negligible in comparison with the second, which accordingly represents the limiting form of $K_1(\alpha)$. Hence

$$K(\alpha) \to K_1(\alpha) \to \frac{2I}{3}\alpha^{-5/6} \quad \text{as} \quad \alpha \to 0.$$

When α is not infinitesimally small, $K(\alpha)$ will differ from its limiting form. It has been estimated that the relative error involved is of the order $\alpha^{1/3}$.

Table I .....

TABLE I.

Theoretical and observed values of T for various depths at which detonators are fired.

Depth (in feet) at which detonator is fired	Total external pressure P (in atmospheres)	Calculated values of T (in milli-seconds)	Observed values of T (in milliseconds)
3	1.09	23.5	23.5, 24.1
25	1.75	15.°	15.2, 16.5, 14.9
45	2.35	12.4	11.4, 11.7
65	2.95	10.3	10.6, 9.2, 9.3
85	3.55	8.75	8.9
105	4.15	7.76	8.4, 7.2
125	4.75	6.93	6.9
145	5.35	6.25	6.4, 6.8
165	5.95	5.73	5.7
185	6.55	5.28	5.5, 5.7
205	7.15	4.92	5.0
225	7.75	4.60	4.8, 5.0
245	8.35	4.33	4.2



FIG. I

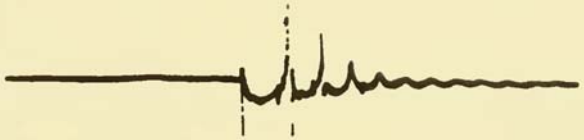


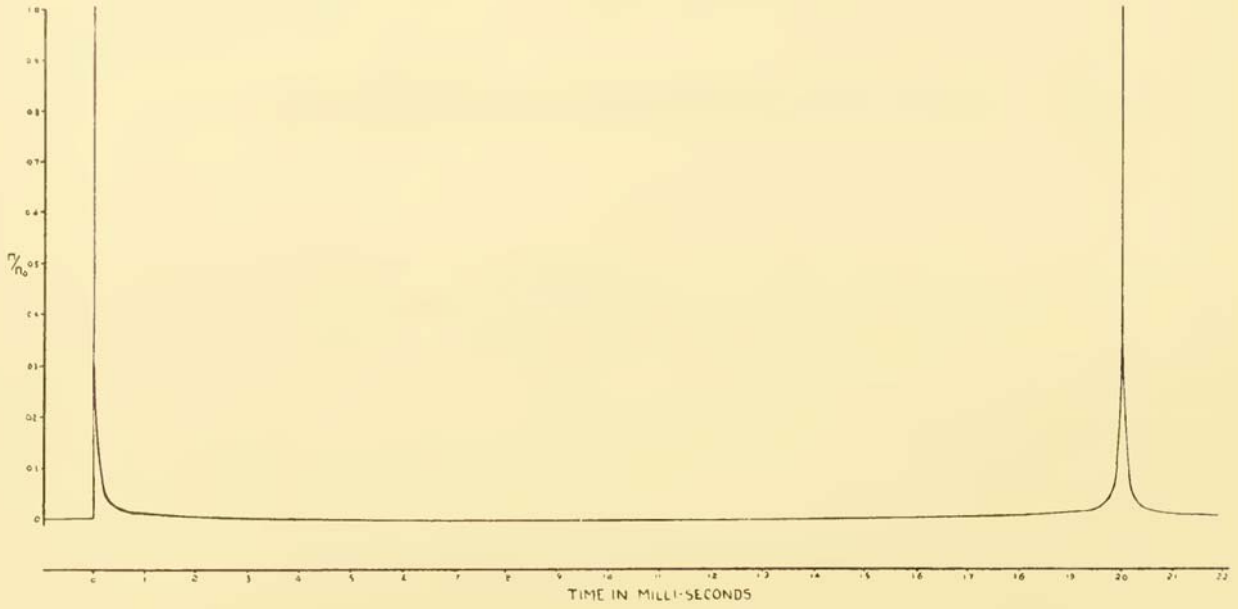
FIG. II



100 Cycles/Sec.

Figure 3.

Calculated form of first two pressure pulses from detonator explosions



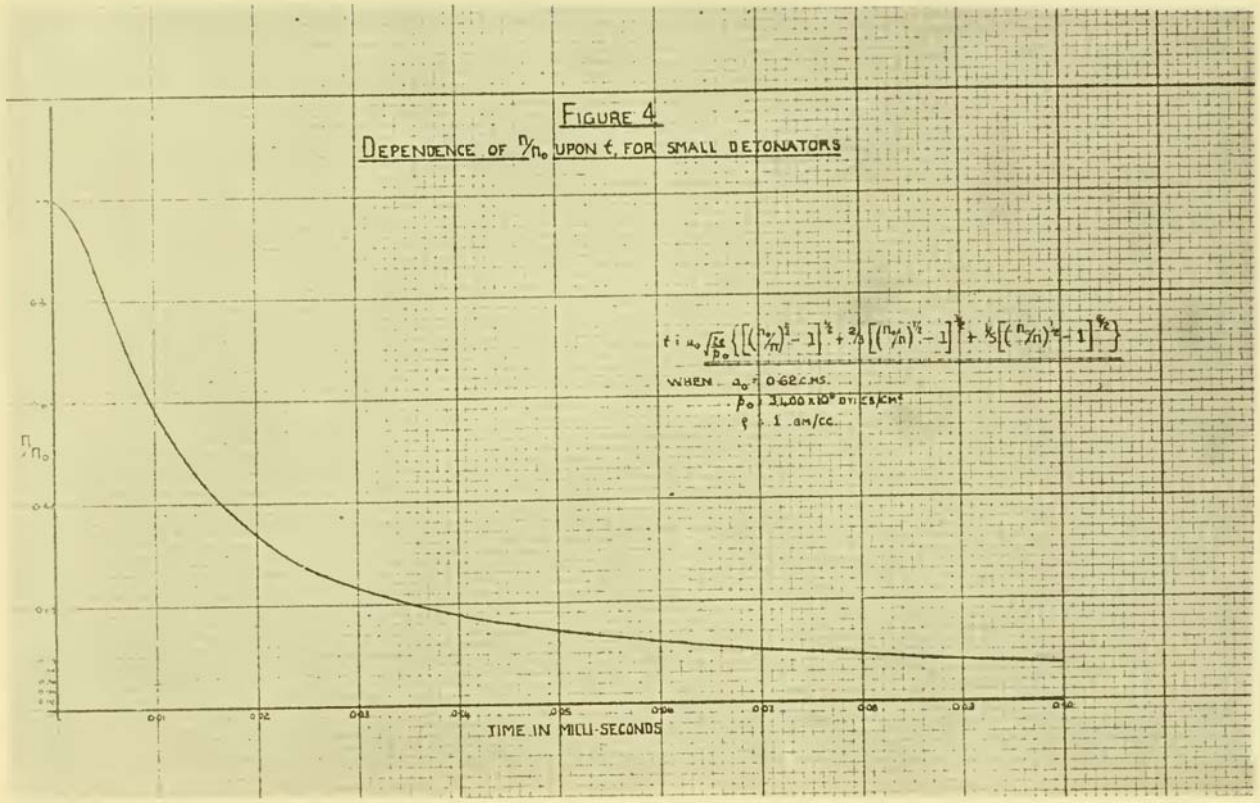


FIGURE 4

DEPENDENCE OF n/n_0 UPON t, FOR SMALL DETONATORS

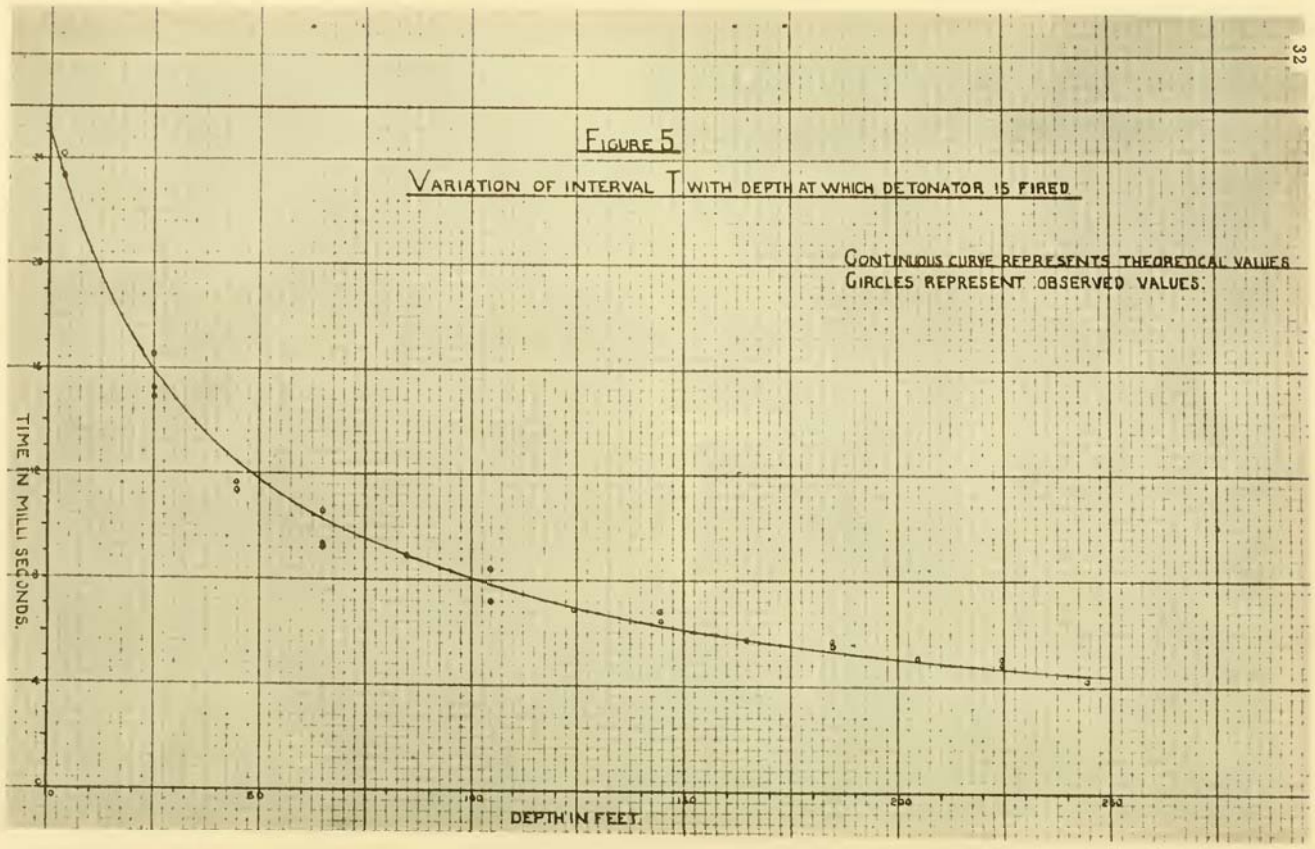


FIGURE 5.

VARIATION OF INTERVAL T WITH DEPTH AT WHICH DETONATOR IS FIRED

CONTINUOUS CURVE REPRESENTS THEORETICAL VALUES
CIRCLES REPRESENT OBSERVED VALUES.

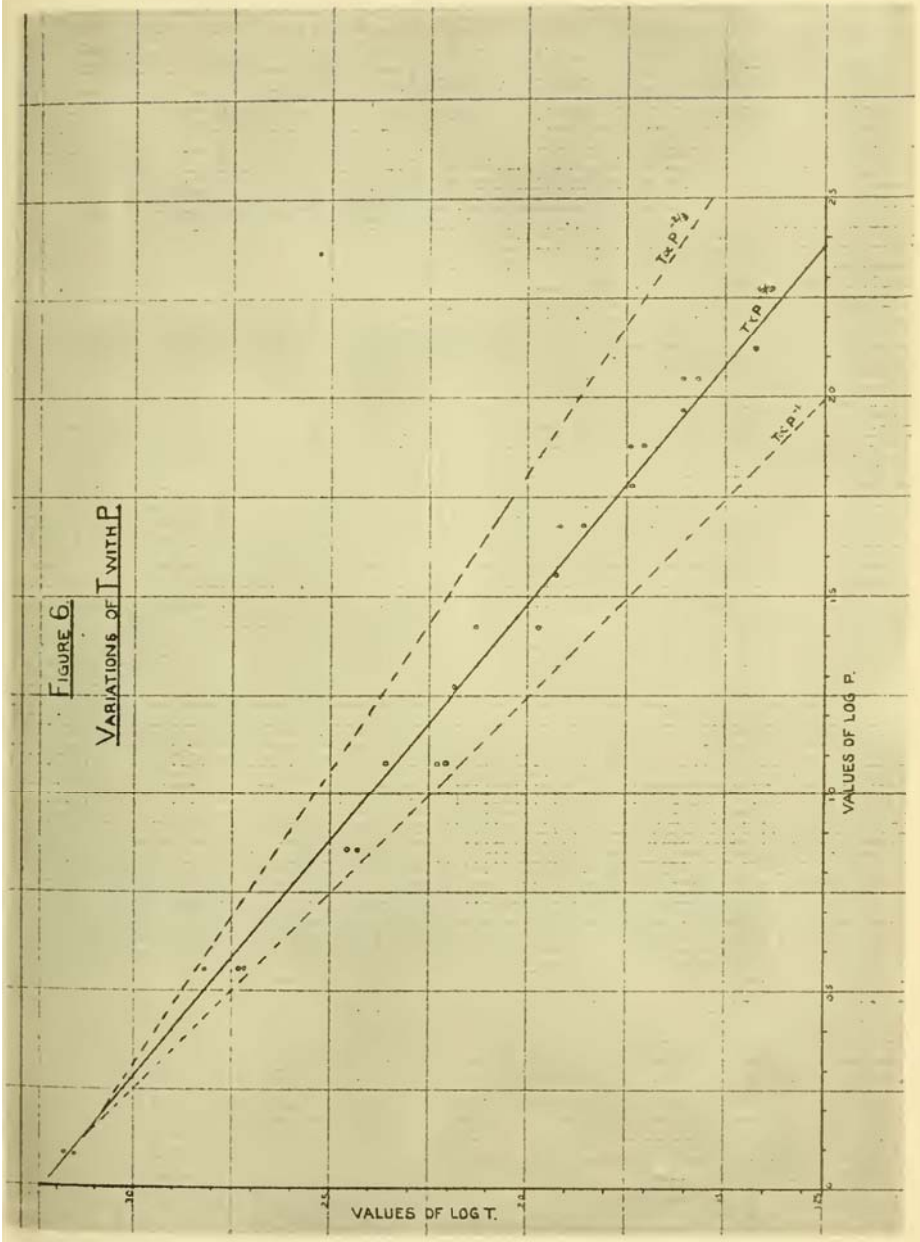


FIGURE 6.
VARIATIONS OF T WITH P.