Geophys. J. R. astr. Soc. (1970) **21,** 137–161.

A Method for Calculating the Output Pressure Waveform from an Air Gun

Anton Ziolkowski

(Received 1970 June 18)

Summary

The signal from an air gun is assumed to behave as if it were derived from an oscillating spherical air bubble in water. The theory of the oscillations of spherical bubbles according to Gilmore is used in conjunction with experimental evidence to derive the complete set of equations necessary to calculate the shape of the radiated pressure waveform anywhere in the water. It is believed that this method will be useful in the design of signal processing techniques and also to improve the design of existing air gun profiling systems.

1. Introduction

Air guns are now widely used in reflection profiling at sea. In commercial prospecting, arrays of guns of different sizes are very often used to produce a broad bandwidth, high-amplitude signal. (B. F. Giles 1968.) Many of the arrangements in use today have been arrived at by trial-and-error methods. An attempt has been made here to produce a method by which the outputs from individual guns can be predicted theoretically if certain parameters of the gun are known. Some ideas on how this method can be applied to the design of arrays of air guns, power spectra and filtering processes are to be discussed elsewhere.

The method is based on the theory of an oscillating spherical bubble, for which some justification is required. The useful band of frequencies of the acoustic radiation emitted by air guns is about 10–100 Hz. The corresponding wavelengths, λ, of this radiation in water are 150–15 m. The radiation is produced by the oscillations of the bubble emitted by the air gun which can expand to perhaps 2-m diameter.

The radiation from an acoustic source of size D is composed of two parts: one whose amplitude falls off as the square of the distance, r, from the source and which dominates at distances less than about D^2/λ; and one whose amplitude decreases with the inverse of the radius and dominates at distances greater than D^2/λ. At the wavelengths with which we are concerned it is only this 'far field' part of the radiation in which we are interested. In the far field of a directive acoustic source the beamwidth of the radiation is of the order of λ/D radians. The useful radiation in the case of air guns must, by this argument, be non-directive, or spherically symmetric. It is therefore reasonable to treat the bubble produced by an air gun like some spherical source, its exact behaviour being determined partly by theory and partly by experiment.

The accuracy of the solution to this problem depends on the various powers of the ratio of bubble-wall velocity, U, to sonic velocity, C, which are neglected. In the zero-order theory, neglecting all terms of order U/C, the water is considered

incompressible. This implies an infinite bulk modulus and thus an infinite speed of sound in the water; there can therefore be no acoustic radiation. The first-order theory, retaining terms of order U/C, considers the bulk modulus to be constant, so that there is a linear dependence of pressure on density, and a constant speed of sound in the water. This is the crudest approach to the problem of acoustic radiation and will only apply when the amplitudes of pressure changes are infinitely small. The second-order theory, accurate to order $(U/C)^2$, is one which allows for finite amplitude oscillations where the particle velocity in the liquid does not approach the speed of sound.

Typically, the bubble produced by an air gun, initially at a pressure of about 20 MNm^{-2} (3000 p.s.i.), may expand from about 0·1 to 1 m in diameter. The amplitudes of oscillation of such a bubble are therefore large and since the particle velocity is at least an order of magnitude less than the sonic velocity, it is the second-order theory that we need.

This theory was first discussed by Gilmore (1952). His paper is an unpublished report, and so, rather than just quoting the final equations, thus giving no idea of how they were derived, a slightly abbreviated form of his argument is given in Section 2.

2. Theory of the oscillations of a spherical bubble (after Gilmore)

If a spherical bubble grows or collapses in an infinite volume of liquid and gravity (see Appendix I) and other asymmetric perturbing effects are neglected, the liquid flow will be spherically symmetric and hence irrotational. In any irrotational flow, the vector quantity, $\mathbf{u}$, can be written in terms of a velocity potential ϕ

$$\mathbf{u} = -\nabla\phi \tag{1}$$

while the equation for the conservation of momentum becomes

$$\frac{\partial}{\partial t}(-\nabla\phi)+(\mathbf{u}.\nabla)\,\mathbf{u} = -\frac{\nabla p}{\rho}+\eta\frac{\nabla^2\mathbf{u}}{\rho}+\left(\frac{\zeta+\frac{1}{3}\eta}{\rho}\right)\nabla(\nabla.\mathbf{u}) \tag{2a}$$

(Equation (15.6), Landau & Lifshitz 1959) where p is the pressure, ρ is the density, η is the viscosity coefficient and ζ is the second viscosity coefficient. In equation (2a) it is assumed that the flow is irrotational and that η and ζ are constant throughout space. η and ζ are normally of the same magnitude. Because of the vector identity

$$\nabla\wedge(\nabla\wedge\mathbf{u}) = \nabla(\nabla.\mathbf{u})-\nabla^2\mathbf{u}$$

where, for irrotational flow, $\nabla\wedge(\nabla\wedge\mathbf{u}) = 0$ we can see that $\nabla(\nabla.\mathbf{u}) = \nabla^2\mathbf{u}$. The last two terms of equation (2a) are therefore of the same order and can be combined using the substitution $\mu = \frac{4}{3}\zeta+\eta$ where μ is the bulk viscosity. If we assume spherical symmetry again the second term in equation (2a) becomes $\nabla(u^2/2)$ and we can rewrite the momentum equation as

$$\frac{\partial}{\partial t}(-\nabla\phi)+\nabla\left(\frac{u^2}{2}\right) = -\frac{\nabla p}{\rho}+\frac{4}{3}\frac{\mu\nabla(\nabla.\mathbf{u})}{\rho}. \tag{2b}$$

The last term of equation (2b) may be transformed by the use of the conservation of mass equation:

$$\nabla.\mathbf{u} = -\frac{1}{\rho}\frac{D\rho}{Dt} \tag{3}$$

where D/Dt is the Lagrangian derivative following the motion of the fluid. It is

evident from equation (3) that the last term of equation (2b) vanishes if either the viscosity or compressibility of the liquid vanishes. We shall be concerned with situations where the viscosity is small, and compressibility moderately small.

The viscous term of equation (2b) is of the order $\mu u/\rho r^2$. This is to be compared with the term $\nabla(u^2/2)$ which is of order u^2/r. Typically we have:

$$u \approx 10\,\mathrm{m\,s^{-1}}, \quad r \approx 10^{-1}\,\mathrm{m}, \quad \mu \approx 10^{-3}\,\mathrm{kg\,m^{-1}\,s^{-1}}, \quad \rho \approx 10^{3}\,\mathrm{kg\,m^{-3}}.$$

Thus the ratio

$$\left(\frac{\mu u}{\rho r^2}\right) \Big/ \left(\frac{u^2}{r}\right) \approx 10^{-6}.$$

It is therefore reasonable to neglect the viscosity term, which is of second-order importance. With these simplifications, equation (2) can be integrated to give the Bernoulli equation:

$$-\frac{\partial \phi}{\partial t} + \frac{u^2}{2} = -\int_{p_\infty}^{p} \frac{dp}{\rho} \tag{4}$$

provided that two assumptions are made: the first assumption is that the pressure p_∞, an infinite distance from the bubble, is constant and the velocity and velocity potential vanish at infinity, so that no constant of integration need appear in equation (4). It is apparent that any deviation from these conditions at infinity would either violate the condition of spherical symmetry or, if spherical, propagate inward and attain infinite amplitude at any finite distance from the bubble, due to spherical convergence. The second assumption is that the liquid density, ρ, can be expressed as a function of pressure only in the radiation we are considering. For isentropic flow, with no heat flow or viscous dissipation, this condition is exactly satisfied. Even if moderately strong thermal and viscous effects are present their effect on the density is usually negligible for liquids because the coefficient of thermal expansion for liquids is so small.

It is convenient to denote the pressure integral of equation (4) by the symbol

$$h(p) = \int_{p_\infty}^{p} \frac{dp}{\rho}. \tag{5}$$

Thermodynamically, the quantity h is the enthalpy difference between the liquid at pressure p and at pressure p_∞.

Under the proper initial and boundary conditions which will be discussed later the flow in the liquid will consist entirely of 'outgoing' velocity and pressure waves. If the velocities were all small compared to sonic velocity, and the sonic velocity did not vary significantly from its constant value, c_∞ at infinity, the well-known expression for diverging spherical sound waves would be applicable

$$\phi = \frac{1}{r} f\left(t - (r/c_\infty)\right), \tag{6}$$

where r is the distance from the centre of the bubble and f is an unspecified function of the argument $(t-(r/c_\infty))$. Equation (4) could then be written

$$r\left(h + \frac{u^2}{2}\right) = \frac{df}{dx}, \tag{7}$$

where $x = t-(r/c_\infty)$. Equations (6) and (7) show that, in the quasi-acoustic (first order) approximation, both the quantities $r\phi$ and $r(h+(u^2/2))$ are propagated outward with a propagation velocity c_∞. As a more accurate approximation when

liquid velocities attain appreciable fractions of the sonic velocity, it is plausible to assume either that $r\phi$ or that $r(h+(u^2/2))$ is propagated outward with a variable velocity $(c+u)$, where c is the local velocity of sound. The alternative assumptions, however, are only approximately equivalent. Kirkwood & Bethe (1942) in their theory of underwater explosions, make the second of these assumptions, i.e. they assume that $r(h+(u^2/2))$ is propagated with a velocity $(c+u)$. If this assumption is also used here then

$$\frac{\partial}{\partial t}\left[r\left(h+\frac{u^2}{2}\right)\right] = -(c+u)\frac{\partial}{\partial r}\left[r\left(h+\frac{u^2}{2}\right)\right]. \tag{8}$$

Equation (8) is more conveniently written in terms of the particle derivative, $D/Dt = \partial/\partial t + u\partial/\partial r$.

$$D/Dt\left[r\left(h+\frac{u^2}{2}\right)\right] + c\frac{\partial}{\partial r}\left[r\left(h+\frac{u^2}{2}\right)\right] = 0, \tag{9}$$

which can be expanded to give:

$$r\frac{Dh}{Dt} + ru\frac{Du}{Dt} + (c+u)\left(h+\frac{u^2}{2}\right) + rc\frac{\partial h}{\partial r} + rcu\frac{\partial u}{\partial r} = 0. \tag{10}$$

In the spherically symmetric situation the momentum relation equation (2) becomes

$$\frac{Du}{Dt} = -\frac{\partial h}{\partial r}, \tag{11}$$

where the viscosity-compressibility interaction term is again neglected; while the continuity relation, equation (3), can be written

$$\frac{\partial u}{\partial r} + \frac{2u}{r} = -\frac{1}{c^2}\frac{Dh}{Dt} \tag{12}$$

since $c^2 = dp/d\rho$ and $dp/\rho = dh$. If derivatives with respect to r are eliminated from equation (10) using equations (11) and (12) a particle derivative relation for spherical flow is obtained.

$$r\frac{Dh}{Dt}\left(1-\frac{u}{c}\right) + ch\left(1+\frac{u}{c}\right) - rc\frac{Du}{Dt}\left(1-\frac{u}{c}\right) - \tfrac{3}{2}cu^2\left(1-\frac{u}{3c}\right) = 0. \tag{13}$$

Equation of state for the liquid

For most liquids it is found experimentally that the pressure-density curve for isentropic compression can be fitted closely by the formula

$$\left(\frac{p+B}{p_\infty+B}\right) = \left(\frac{\rho}{\rho_\infty}\right)^n, \tag{14}$$

where B and n are constants which depend upon the particular liquid under consideration. By expressing p in terms of ρ using a Taylor series (expansion about ρ_∞), it can be seen from the linear term that $K = n(p_\infty+B)$ where K is the bulk modulus. Gilmore suggests that $n \approx 7$ and $B \approx 3000$ atmospheres for water. Figures given in Kaye & Laby (1957) for the compressibility of water show that $n = 8$, $B = 2500$

atmospheres would be better. From equation (14) it is found that

$$c^2 = \frac{dp}{d\rho} = \frac{n(p+B)}{\rho} = \frac{n(p+B)}{(n-1)\rho_\infty}\left(\frac{p+B}{p_\infty+B}\right)^{-(1/n)} \tag{15}$$

and hence

$$c = c_\infty\left(\frac{p+B}{p_\infty+B}\right)^{(n-1)/2n} \tag{16}$$

h can be evaluated with the help of equations (5) and (14).

$$h = \int_{p_\infty}^{p}\left(\frac{p+B}{p_\infty+B}\right)^{-(1/n)}\frac{dp}{\rho_\infty} = \frac{n(p_\infty+B)}{(n-1)\rho_\infty}\left[\left(\frac{p+B}{p_\infty+B}\right)^{(n-1)/n}-1\right]. \tag{17}$$

Bubble wall motion and boundary conditions

The hydrodynamic analysis has been carried out using Lagrangian co-ordinates following fluid particles. Since the growing or collapsing bubble wall is a 'particle path', we may apply the boundary conditions at the bubble wall and thus determine the variation of bubble radius with time. Let the capital letters R, U, C, H denote the values which the previously defined parameters r, u, c, h assume in the liquid at the bubble wall and designate time derivatives by dots. Then applying equation (13) at the bubble wall and dividing by C, we have:

$$R\dot{U}\left(1-\frac{U}{C}\right)+\tfrac{3}{2}U^2\left(1-\frac{U}{3C}\right) = H\left(1+\frac{U}{C}\right)+\frac{R\dot{H}}{C}\left(1-\frac{U}{C}\right), \tag{18}$$

which is the equation of motion of the bubble wall. In this equation H and C are given by:

$$H = \frac{n(p_\infty+B)}{(n-1)\rho_\infty}\left[\left(\frac{P+B}{p_\infty+B}\right)^{(n-1)/n}-1\right], \tag{19}$$

$$C = c_\infty\left(\frac{P+B}{p_\infty+B}\right)^{(n-1)/2n}, \tag{20}$$

where P is the pressure in the liquid at the bubble wall. In order to solve equation (18) it is necessary to know P as a function of R.

If the liquid has an appreciable viscosity, μ, or surface tension, σ, the pressure P in the liquid at the bubble wall does not equal the pressure Pi exerted on the bubble wall by any interior gas or vapour, but the two pressures are related by the equation:

$$P = Pi-\frac{2\sigma}{R}+2\mu\left(\frac{\partial u}{\partial r}\right)_{r=R}. \tag{21}$$

The last term on the right-hand side of equation (21) is small if either the viscosity or compressibility is small and it can be shown to be of the same order of magnitude as the viscous term neglected in equation (2b). The surface tension term can equally easily be shown to be negligible for bubbles of this size and at these pressures so that

$$P = Pi. \tag{22}$$

The internal pressure Pi can be split up into two terms P_g and P_v which represent the partial pressures of the gas and water vapour, respectively. Thus we have

$$P = P_g+P_v. \tag{23}$$

Velocity and pressure fields throughout the liquid

A. *First-order approximation.* We begin by considering the simple quasi-acoustic equations as this work serves to guide the investigation of higher-order approximations. In the quasi-acoustic approximation, accurate to terms of the first order in u/c_∞, the radial velocity is found by combining equations (1) and (6):

$$u = \frac{f(t-(r/c_\infty))}{r^2} + \frac{f'(t-(r/c_\infty))}{rc_\infty}, \tag{24}$$

where f' is the total differential of the function f with respect to the argument $(t-(r/c_\infty))$.

With the help of the first-order approximation $h = (p-p_\infty)/\rho_\infty$ equation (7) can be written

$$p = p_\infty + \frac{\rho_\infty}{r} \cdot f'(t-(r/c_\infty)) - \frac{\rho_\infty u^2}{2}. \tag{25}$$

When equations (24) and (25) are solved simultaneously for f and f' the results are

$$f(t-(r/c_\infty)) = \frac{r^2}{c_\infty}\left(c_\infty u - \frac{u^2}{2} - \frac{p-p_\infty}{\rho_\infty}\right), \tag{26}$$

$$f'(t-(r/c_\infty)) = r\left(\frac{u^2}{2} + \frac{p-p_\infty}{\rho_\infty}\right), \tag{27}$$

from which we may deduce the following theorem.

In the quasi-acoustic approximation the two quantities

(a) $$r\left(c_\infty u - \frac{u^2}{2} - \frac{p-p_\infty}{\rho_\infty}\right)$$

and

(b) $$\left(\frac{u^2}{2} + \frac{p-p_\infty}{\rho_\infty}\right)$$

are propagated outwards with a velocity c_∞ and magnitudes decreasing inversely with the radius.

If u and p are calculated by the method of characteristics for a point far enough from the bubble wall to be out of the high-velocity region the two quantities (a) and (b) can be readily calculated for this point and then extended to all other points further from the bubble using the above theorem. Having calculated these quantities throughout the region of interest, one may readily solve for u and p throughout the region.

B. *Second-order approximation.* The method described above is accurate only to first-order terms in u/c_∞. This suggests a method by which second-order approximations may be made using the Kirkwood–Bethe hypothesis. This states that a quantity which will be denoted by y,

$$y(r, t) = r\left(h + \frac{u^2}{2}\right) \tag{28}$$

is constant along any path traced by a point moving outward with the variable velocity $c+u$. Such a path is known as an 'outgoing characteristic'. If differentiation is denoted by d/dr where

$$\frac{d}{dr} = \frac{\partial}{\partial r} + \frac{1}{c+u}\frac{\partial}{\partial t}, \tag{29}$$

the Kirkwood–Bethe hypothesis gives

$$\frac{dy}{dr} = 0. \tag{30}$$

In the quasi-acoustic approximation the quantity

$$z(r, t) = c_\infty r^2 u - ry \tag{31}$$

also remains constant along an outgoing characteristic (see (a)). For a second-order approximation, the variation of z along the characteristic should be determined approximately. Differentiation of equation (31) gives

$$\frac{dz}{dr} = 2c_\infty ru + c_\infty r^2 \frac{du}{dr} - y, \tag{32}$$

since $dy/dr = 0$. The derivative du/dr may be evaluated from the basic continuity and momentum equations. When equation (12) is multiplied by c and added to equation (11), the result can be written

$$(c+u)\frac{du}{dr} = -\frac{2cu}{r} - \frac{c+u}{c}\frac{dh}{dr}, \tag{33}$$

which is equivalent to a well-known formula for integration along a characteristic. Upon solving equation (28) for h, introducing this value into equation (33) and solving for du/dr one obtains

$$\frac{du}{dr} = -\frac{2c^2 u}{(c^2-u^2)r} + \frac{y}{(c-u)r^2}. \tag{34}$$

Substitution of this result into equation (32) yields

$$\frac{dz}{dr} = -\frac{2c_\infty ru^3}{c^2-u^2} + \frac{(c_\infty - c+u)}{c-u}y. \tag{35}$$

As expected the relative rate of variation of z with r is small, being of the order of $(u/c)^2$ from equation (35). It is permissible, therefore, to use the lowest-order approximations to the terms on the right-hand side of equation (35). It can be shown that $(c_\infty - c)/c$ is of the order of $(u/c)^2$ and so the approximation becomes:

$$\frac{dz}{dr} = -\frac{2u^3 r}{c_\infty} + \frac{uy}{c_\infty}. \tag{36}$$

From equation (31) $u \approx z/c_\infty r^2$. With this substitution equation (36) becomes

$$\frac{dz}{dr} = -\frac{2z^3}{c_\infty{}^4 r^5} + \frac{yz}{c_\infty{}^2 r^2}. \tag{37}$$

Although equation (37) is non-linear, it can be made linear by taking $1/z^2$ as a new dependent variable. The solution is then found by standard methods to be

$$z = \frac{y^2}{c_\infty{}^2}\left[\frac{3}{2} + \frac{3y}{c_\infty{}^2 r} + \frac{3y^2}{c_\infty{}^4 r^2} + \frac{2y^3}{c_\infty{}^6 r^3} + K_1 \exp\left(\frac{2y}{c_\infty{}^2 r}\right)\right]^{-\frac{1}{2}}, \tag{38}$$

where K_1 is an arbitrary constant.

The exponential in equation (38) can be expanded in a power series to give

$$z = \frac{y^2}{c_\infty{}^2}\left[K_2\left(1+\frac{2y}{c_\infty{}^2 r} + \frac{2y^2}{c_\infty{}^4 r^2} + \frac{4y^3}{3c_\infty{}^6 r^3}\right) + (K_2 - \tfrac{3}{2})\left(\frac{2y^4}{3c_\infty{}^8 r^4} + \ldots\right)\right]^{-\frac{1}{2}}, \tag{39}$$

where $K_2 = K_1 + \frac{3}{2}$. It can be seen from equation (28) that $y/c_\infty{}^2 r$ is of the order of $u^2/c_\infty{}^2$. It can also be seen from equation (31) that $c_\infty{}^2 z/y^2$ is of the order $c_\infty{}^3/u^3$. It follows from equation (39) that K_2 is of the order u^6/c^6 and thus the approximation

$$z = \frac{y^2}{c_\infty{}^2}\left[K_2\left(1+\frac{2y}{c_\infty{}^2 r}\right) - \frac{y^4}{c_\infty{}^8 r^4}\right]^{-\frac{1}{2}} \approx \frac{K_3 y^2}{c_\infty{}^3}\left(1 - \frac{y}{c_\infty{}^2 r} + \frac{K_3{}^2 y^4}{2c_\infty{}^8 r^4}\right), \tag{40}$$

where $K_3 = K_2{}^{-\frac{1}{2}}$ has a negligible relative error (of order $(u/c)^4$). The velocity may then be obtained from equations (31) and (40)

$$u = \frac{y}{c_\infty r} + \frac{K_3 y^2}{c_\infty{}^3 r^2}\left(1 - \frac{y}{c_\infty{}^2 r} + \frac{K_3{}^2 y^4}{2c_\infty{}^8 r^4}\right). \tag{41}$$

Using the second-order approximation for p

$$p - p_\infty \approx \rho_\infty h + \frac{\rho_\infty h^2}{2c^2} \tag{42}$$

and substituting for h from equation (28) we have

$$p - p_\infty = \rho_\infty\left(\frac{y}{r} - \frac{u^2}{2}\right) + \frac{\rho_\infty}{2c_\infty{}^2}\left(\frac{y}{r} - \frac{u^2}{2}\right)^2. \tag{43}$$

Along any particular outgoing characteristic, the quantities y and K_3 are constants which may be determined if the velocity and pressure at one point along the characteristic are known. For example, if $U(R)$ and $P(R)$ at the bubble wall are determined, for a given instant, the value of y along the corresponding outgoing characteristic is found, with the help of equation (28), to be:

$$y = R\left(H + \frac{U^2}{2}\right). \tag{44}$$

The value of K_3 is determined by putting $u = U$ and $r = R$ in equation (41). A first approximation, $K_3 = c_\infty{}^3 R^2 U/y^2$, is found by neglecting all but the lowest-order terms. This value for K_3 is then inserted into the small term $K_3{}^2 y^4/2c_\infty{}^8 R^4$, and the resulting equation solved to obtain a value of K_3 accurate to second order.

$$K_3 = \frac{c_\infty{}^3 R^2 U}{y^2}\left(1 - \frac{U^2}{2c_\infty{}^2}\right) - \frac{c_\infty{}^2 R}{y}\left(1 - \frac{U}{c_\infty}\right). \tag{45}$$

Equations (41) and (43) with constants evaluated from equations (44) and (45) yield values for the velocity and pressure along any outgoing characteristic as a function of r.

3. Pressure-radius relation, initial conditions and damping factors for prediction of wave forms

In order to use the equations in Section 2 to determine the radiated pressure wave form from a spherical oscillating air bubble in water, we need a relation between the pressure in the water at the bubble wall, P, and the radius of the bubble, R. The relation between R and the partial pressure of the air, Pg, is assumed to be

$$Pg = p_\infty \left(\frac{Ro}{R}\right)^b \tag{46}$$

(Equation (11), H. G. Flynn 1964), where p_∞ is the pressure in the water at an infinite distance from the bubble, and is constant; Ro is the radius of the bubble at which the partial pressure of the air, Pg, is equal to p_∞; b is, as yet, unknown.

This can easily be recognized as the equation for a fixed mass of gas:

$$pV^n = \text{constant},$$

where $V = \frac{4}{3}\pi R^3$ and therefore $b = 3n$.

From thermodynamics we know that $n = 1{\cdot}0$ is the condition for isothermal expansions and compressions, when there is perfect heat transfer between the air and water. When $n = \gamma = Cp/Cv = 1{\cdot}4$ (for air) this is the condition for perfect adiabatic oscillation, with no heat transfer between the water and air. Clearly the present situation must lie somewhere between these two limits, i.e. $3{\cdot}0 < b < 4{\cdot}2$.

If we substitute for Pg from equation (46) into equation (23) and neglect P_v we have

$$P = Pg = p_\infty \left(\frac{Ro}{R}\right)^b. \tag{47}$$

The vapour pressure in the bubble, P_v, can attain its maximum value when the partial pressure of the air, Pg, is a minimum. This minimum occurs when the bubble radius is a maximum. Under normal working conditions it is found that R does not exceed $2Ro$ and that $b \approx 3{\cdot}4$ (see below). If we take these values for R and b and assume that $p_\infty \approx 2$ atm (at a working depth of 14 m) we find from equation (46) that

$$(Pg)_{\min} \approx 2 \times (0{\cdot}5)^{3{\cdot}4} \approx 0{\cdot}2 \text{ atmospheres.}$$

The partial pressure of the water vapour can never exceed 10 per cent of this value even if the water is warm and the saturation vapour pressure is reached. It therefore seems reasonable to neglect P_v in equation (23). We now have all the equations necessary to determine the radiated pressure wave from such a bubble, at some range r, which may be solved by an iterative numerical integration procedure. They are presented here, in order, for clarity, together with four new equations (48), (49), (50) and (51) which relate the bubble wall velocity and radius increments, ΔU and ΔR, to the time increment Δt, at each iteration.

$$P = p_\infty \left(\frac{Ro}{R}\right)^b \tag{47}$$

$$H = \frac{n(p_\infty + B)}{(n-1)\,\rho_\infty}\left[\left(\frac{P+B}{p_\infty + B}\right)^{(n-1)/n} - 1\right] \tag{19}$$

$$C = c_\infty \left(\frac{P+B}{p_\infty + B}\right)^{(n-1)/2n} \tag{20}$$

$$\frac{dU}{dt} = \frac{H(1+(U/C))+R/C(dH/dt)(1-(U/C))-\frac{3}{2}U^2(1-(U/3C))}{R(1-(U/C))} \tag{18}$$

$$\Delta U = \frac{dU}{dt} \cdot \Delta t + \frac{d^2U}{dt^2} \cdot \frac{\Delta t^2}{2} \tag{48}$$

$$U_{new} = U + \Delta U \tag{49}$$

$$\Delta R = U\Delta t + \frac{dU}{dt} \cdot \frac{\Delta t^2}{2} \tag{50}$$

$$R_{new} = R + \Delta R \tag{51}$$

$$y = R(H+(U^2/2)) \tag{44}$$

$$K_3 = \frac{c_\infty{}^3 R^2 U}{y^2}\left(1-\frac{U^2}{2c_\infty{}^2}\right) - \frac{c_\infty{}^2 R}{y}\left(1-\frac{U}{c_\infty}\right) \tag{45}$$

$$u = \frac{y}{c_\infty r} + \frac{K_3 y^2}{c_\infty{}^3 r^2}\left(1-\frac{y}{c_\infty{}^2 r}+\frac{K_3{}^2 y^4}{2c_\infty{}^8 r^4}\right) \tag{41}$$

$$p-p_\infty = \rho_\infty\left(\frac{y}{r}-\frac{u^2}{2}\right)+\frac{\rho_\infty}{2c_\infty{}^2}\left(\frac{y}{r}-\frac{u^2}{2}\right)^2 \tag{43}$$

Time derivatives such as dH/dt in equation (18) may be found using the relation $d/dt = Ud/dR$. The second-order term in equation (48) is needed at times when the particle velocity, U, at the bubble wall, is close to a stationary value. This term can be obtained by differentiating equation (18). All the time derivatives occurring in the resulting differential may be found using equations (47), (18), (19) and (20). They are not reproduced here as they are rather lengthy expressions. The need for the second-order term in equation (50) arises in the same way as that in equation (48).

A value for Δt must be chosen to be suitably small compared with the period, T, of the bubble oscillation. Typically, T is of the order of 10–100 ms. It has been found by experience that $\Delta t = 10^{-4}$ s is too large and that $\Delta t = 10^{-5}$ gives the same answers to within four significant figures as $\Delta t = 10^{-6}$ s after 100 ms of oscillation. Therefore $\Delta t = 10^{-5}$ s is probably good enough for most purposes.

Initial conditions

The initial conditions which have been used are the simplest possible and have been discussed by Gilmore. All the air at the instant of firing is assumed to be transferred instantaneously from the gun to a spherical bubble of the same volume and pressure as the gun (see Appendix II). The parameters of the gun that we know are its volume, V, depth, HL, and gauge pressure on deck at the instant of firing, X. We also know atmospheric pressure p_0.

If we denote our initial starting pressure, bubble radius and bubble wall velocity by P_{IN}, R_{IN} and U_{IN} respectively, we have:

$$P_{IN} = X + p_0 \tag{52}$$

$$R_{IN} = (3V/4\pi)^{\frac{1}{3}} \tag{53}$$

$$p_\infty = p_0 + \rho_\infty gHL \tag{54}$$

$$U_{IN} = 0. \tag{55}$$

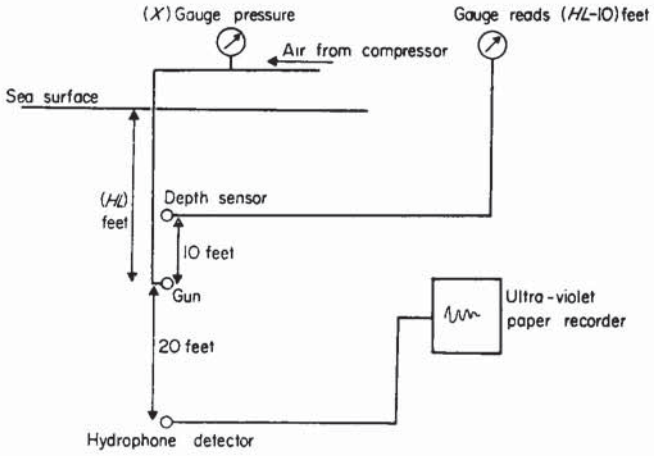


FIG. 1. Diagram to show the geometry of the air gun waveform measurements; the depth sensor, gun and hydrophone detector were in a vertical line.

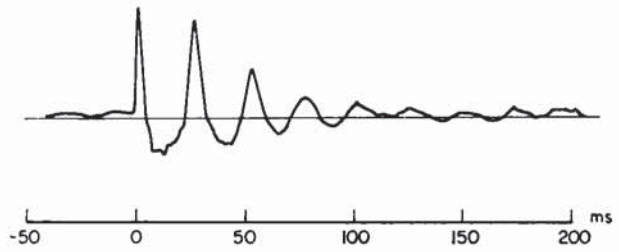


FIG. 2. Waveform recorded with the system shown in Fig. 1. Volume of gun (V) = 0·155 l (9·4 cubic inches); gauge pressure of gun (X) = 13·3 MNm^{-2} (1940 p.s.i.); depth of gun (HL) = 14 m (46 ft).

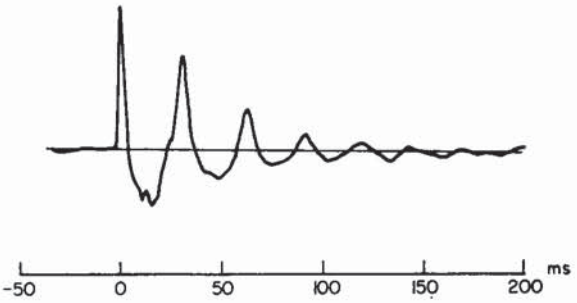


FIG. 3. Waveform recorded with the system shown in Fig. 1. Volume of gun (V) = 0·5 l (30·5 cubic inches); gauge pressure of gun (X) = 10·6 MNm^{-2} (1540 p.s.i.); depth of gun (HL) = 18·3 m (60 ft).

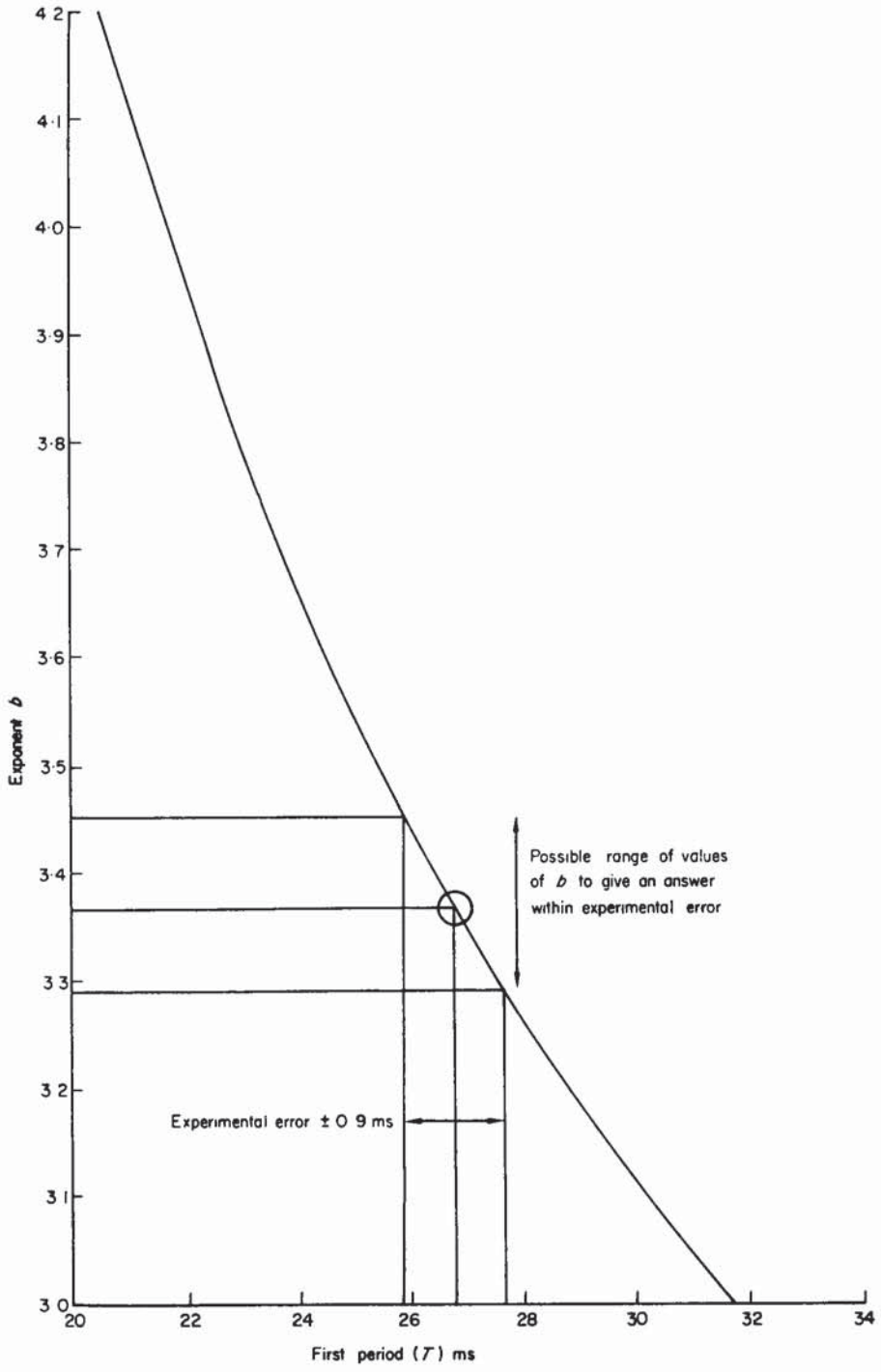


FIG. 4. Graph of variation of predicted first period of oscillation as exponent b is varied. The following data were used: $V = 0{\cdot}155$ l; $X = 13{\cdot}3$ MNm^{-2}; $HL = 14$ m.

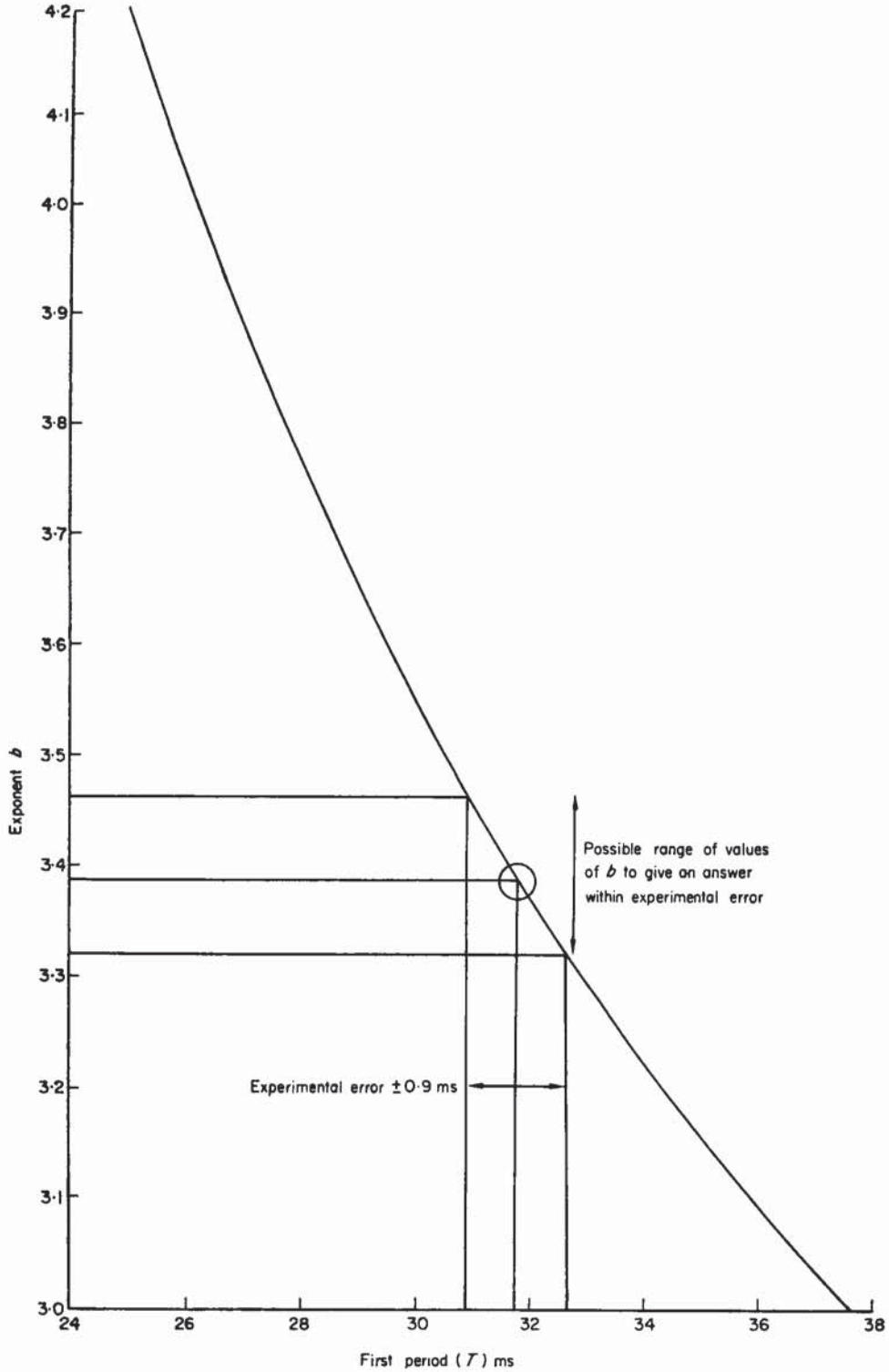


FIG. 5. Graph of variation of predicted first period of oscillation as exponent b is varied. The following data were used: $V = 0{\cdot}5$ l; $X = 10{\cdot}6\ \mathrm{MNm^{-2}}$; $HL = 18{\cdot}3$ m.

Gilmore has pointed out that the pressure discontinuity at the bubble wall, $(P_{IN}-p_0)$, in these conditions, produces a velocity jump. Therefore in equation (18) the terms in dU/dt and dH/dt are infinitely greater than the other terms for the first instant. Therefore, initially,

$$\frac{RdU}{dt}\left(1-(U/C)\right) = \frac{R}{C}\frac{dH}{dt}\left(1-(U/C)\right). \tag{56}$$

Cancelling common factors and integrating:

$$U_{IN}+ = \int \frac{dH}{C} = \frac{H}{c_\infty}$$

to first order. We may, therefore, use this value as our initial U in the first instance of the use of equation (18).

For any set of input data (V, HL, X, p_0) the only parameter which can be adjusted to reconcile theory with experiment is the exponent b in equation (47).

Experimental determination of b

It is found for any set of input data that the effect of altering the value of b is to change the period of oscillation. However, the period itself decreases as the oscillation progresses and decays in amplitude, since the bubble is losing acoustic energy to the water. To determine a value of b under a given set of starting conditions, therefore, a value of b has been chosen which gives the same first period of oscillation in the case of the predicted bubble waveform as in the case of the real waveform obtained with the arrangement represented in Fig. 1.

Two pressure waveforms were taken from guns of 0·1551 (9·5 in^3) and 0·501 (30·5 in^3). These are shown in Figs 2 and 3. The first periods as measured between the first two pressure peaks on these waveforms are 26·8 and 31·8 ms, respectively. The graphs of b versus T (first period of oscillation) obtained with the above theory, for these two sets of data, are shown in Figs 4 and 5.

The error in the estimation of the period (see Appendix II) is 0·9 ms; this gives a range of b's (3·285–3·450) and (3·315–3·465), respectively, with a large overlap of values between the two cases. This leads one to suspect that one value of b might be sufficiently good for use with both guns. Tables 1 and 2 show a comparison of the first periods obtained experimentally with those computed using this theory and a value for b of 3·4.

Table 1

Gun size (V) = 9·4 *cubic inches*

b = 3·4

Gauge pressure (X) p.s.i.g.	Depth (HL) ft	First period Theory (Tth) ms	Measurement (Tm) ms	Error (Tm–Tth)
2070·0	10·0	44·95	45·5	+0·6
2110·0	20·0	37·95	38·6	+0·7
2100·0	27·0	34·15	34·0	−0·2
2180·0	37·0	30·37	30·8	+0·4
1940·0	46·0	26·43	26·4	0·0
1960·0	55·0	24·11	25·0	+0·9
1970·0	64·5	22·27	23·0	+0·7
1920·0	76·0	20·12	20·7	+0·6

Table 2

Gun size (V) = 30·5 *cubic inches*

b = 3·4

Gauge pressure (X) p.s.i.g.	Depth (HL) ft	First period Theory (Tth) ms	First period Measurement (Tm) ms	Error (Tm–Tth)
1580·0	27·0	46·10	45·3	−0·8
1580·0	38·0	40·00	38·5	−1·5
1550·0	48·0	35·60	34·0	−1·6
1540·0	60·0	31·68	31·8	+0·1
1510·0	71·0	28·68	28·5	−0·2
1490·0	81·0	26·45	26·2	−0·3
1480·0	92·0	24·45	23·8	−0·7
1460·0	103·0	22·69	22·2	−0·5

All the values in Table 1 and all but two in Table 2 are within experimental error. Agreement between theory and experiment can be made closer by choosing a slightly smaller value of b for the smaller gun (e.g. b = 3·36) and a slightly larger value for the larger gun (e.g. b = 3·42). All estimates are then within experimental error.

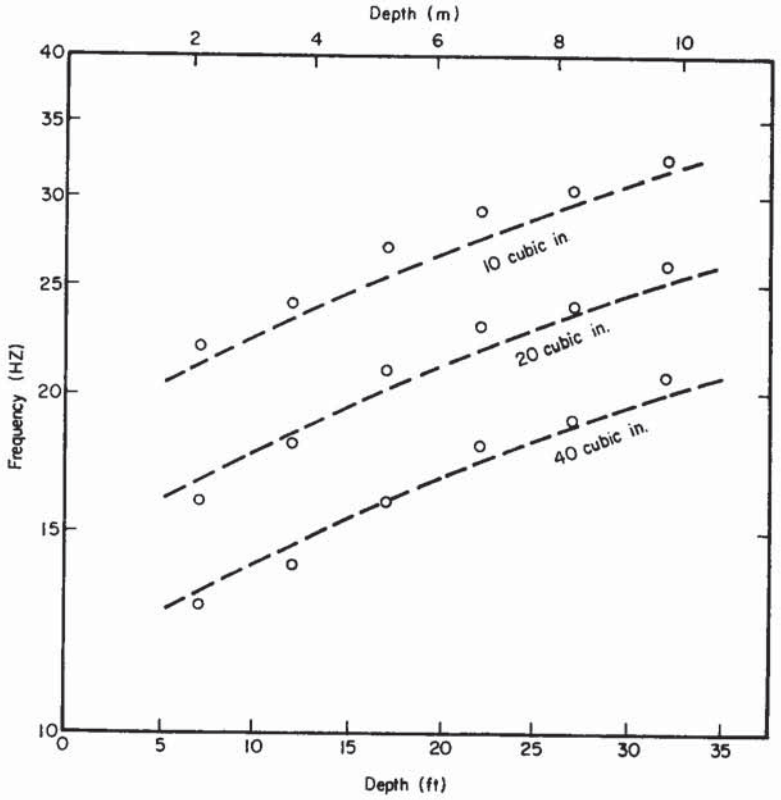


FIG. 6. Graph of bubble frequency as a function of depth for gun volumes 10, 20, 40 cubic inches. The frequency as calculated is T^{-1} where T is the first period of oscillation in seconds. Giles's results are circles; theoretical results are dotted lines. All guns are operating at 1900 p.s.i.g. b = 3·4.

It is concluded from this evidence that the theory is sufficiently good to predict the first period of oscillation for a particular gun of this design (Lamont type: Ewing & Zaunere 1964) to within experimental error by matching the value of b to any given output waveform from that gun. However, if none is available a value of $b = 3{\cdot}4$ will give answers accurate to about 1 ms for guns of this sort of size. Predicted graphs (Fig. 6) using this theory can be seen to be in very close agreement with results obtained experimentally by Giles (1968) using guns of entirely different design.

The main power of this theory, however, is in predicting the pressure waveform emitted by the guns.

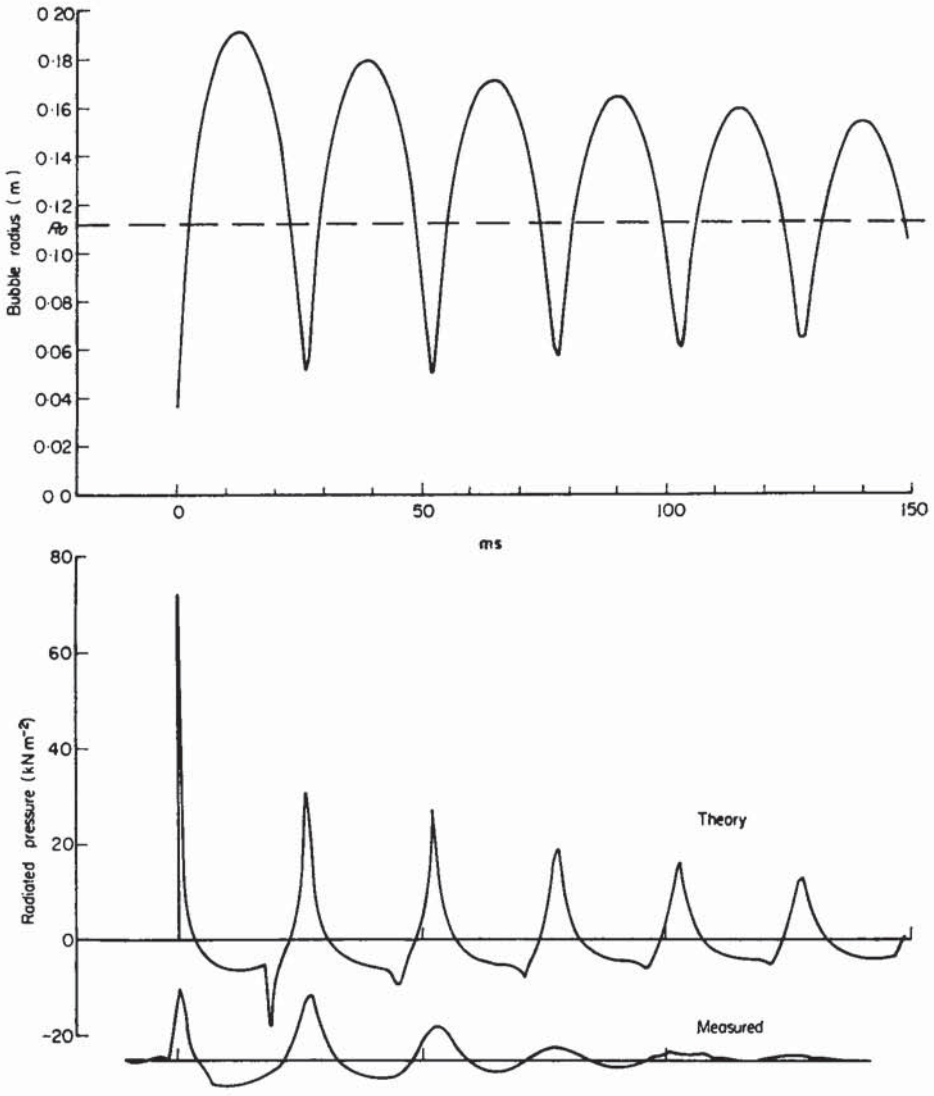


FIG. 7. A comparison of theoretical and measured waveforms for the 0·155 l gun. $X = 13{\cdot}3$ MNm^{-2}, $HL = 14$ m. Ro is the radius of the bubble at which the internal pressure and hydrostatic pressure are equal. A value for b of 3·4 was used.

Waveform matching and damping factors

Attempts at matching theoretical waveforms with those obtained experimentally have shown that the oscillations of the real bubble decay faster than those of the theoretical one. (See Figs 7 and 8.) An explanation for this can be found by considering equation (47).

In the earlier discussion of the thermodynamic significance of the exponent b, it was noted that its value was related to the heat transfer between the air in the bubble and the water. It can easily be shown by order of magnitude arguments that there is insufficient time for any appreciable heat transfer to take place between the air and the water, over one period of oscillation. Therefore one would expect the value of b to be close to 4·2. As it is not, some of our assumptions must be wrong.

The assumption that we are dealing with a spherical bubble at all times is clearly not tenable. Perturbing effects such as gravity, the proximity of the gun and towing apparatus, and the fact that the bubble was not initially a sphere, will all cause deformations of the spherical bubble which we are considering. The bubble may even be inherently unstable. All these deformations will probably cause turbulence, perhaps even cavitation in the water, and will, in any case, use up some of the initial energy at our disposal.

The existence of such energy losses from spherical bubbles, apart from acoustic radiation, has been known for a long time (Herring 1949).

As long as the bulk of the air remains in only one bubble, we can deal with a fixed mass of air and allow for these local energy losses by multiplying our bubble time-varying functions by some decaying function of time. This can be done in the following way:

The function we want to damp is $R(t)$, as this will then have a suitable effect on all the other time-dependent quantities. As the radius is always positive, but must, in the steady state condition, satisfy the equation $R(\infty) = Ro$, we can apply $e^{-\alpha t}$ about a zero taken at $R = Ro$. This can be expressed as

$$R'(t) = \big(R(t) - Ro\big)\, e^{-\alpha t} + Ro \tag{57}$$

where $R(t)$ is the old radius—time function which we have just computed, and $R'(t)$ is the new, damped function, and α is the damping variable. (α can be considered constant; see below.) This equation satisfies our boundary conditions that $R'(o) = R(o) = R_{IN}$ and $R'(\infty) = Ro$.

The quantities R, U and H are the only ones which will be affected by this artificial decay in predicting the radiated pressure wave from equations (44), (45), (41) and (43). The new H can be found using the equations

$$P' = p_\infty \left(\frac{Ro}{R'}\right)^b, \tag{58}$$

$$H' = \frac{n(p_\infty + B)}{(n-1)\,\rho_\infty} \left[\left(\frac{P' + B}{p_\infty + B}\right)^{(n-1)/n} - 1\right]. \tag{59}$$

The new U can be found by differentiating equation (57) with the result:

$$\frac{dR'}{dt} = U'(t) = e^{-\alpha t}\left[U(t) - \alpha\big(R(t) - Ro\big)\right], \tag{60}$$

where $U(t)$ is the old function of time ($U = dR/dt$) and $U'(t)$ is the new, damped function.

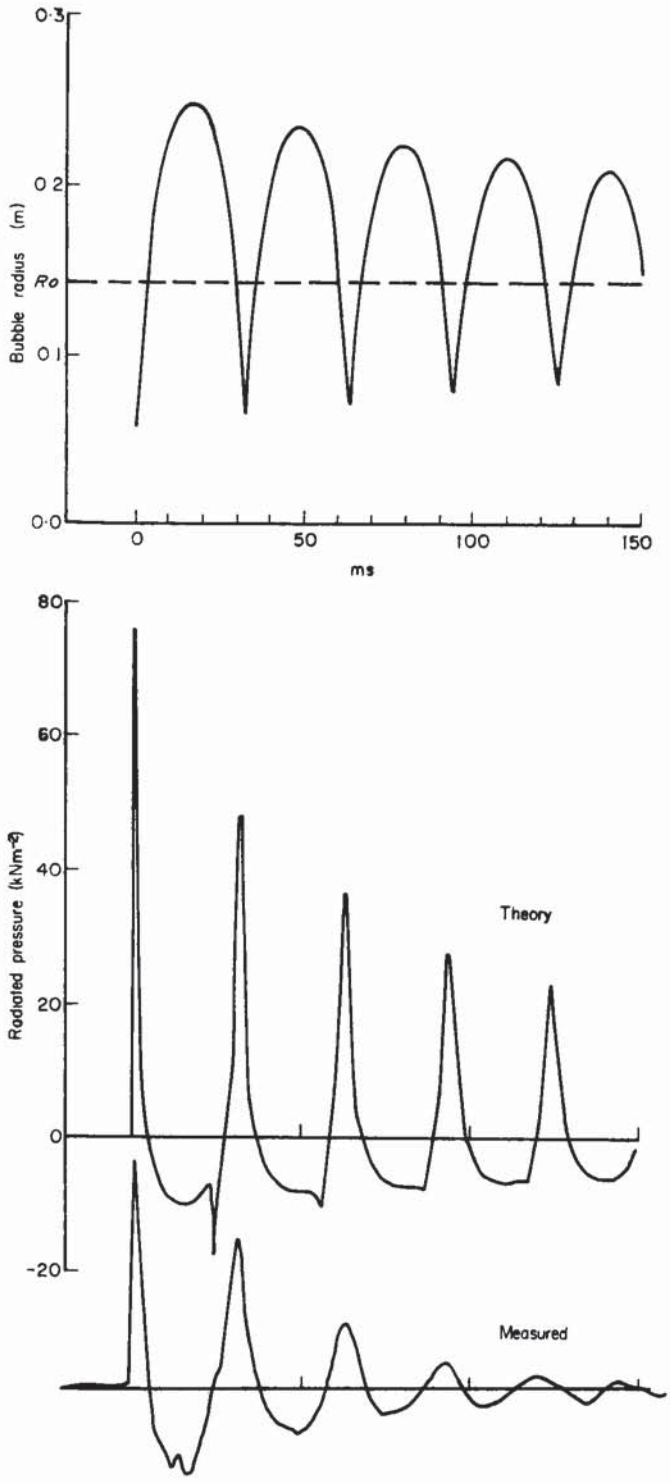


FIG. 8. A comparison of theoretical and measured waveforms for the 0·5 l gun. $X = 10{\cdot}6\ \mathrm{MNm}^{-2}$, $HL = 18{\cdot}3$ m. *Ro* is the radius of the bubble at which the internal pressure and hydrostatic pressure are equal. A value for b of 3·4 was used.

With these additional equations ((57)–(60)) we are now able to allow for the perturbing effects and can substitute the results into equations (44), (45), (41) and (43) to obtain the damped waveforms we require. The results obtained for the 0·1551 and 0·51 guns are shown in Figs 9 and 10. A value for α of 11·0 has been used in both cases.

These waveforms are calculated assuming a reflection coefficient at the sea surface of -1. The waveform at the detector is then composed of a direct wave which has travelled a distance $D1$ and a reflected wave which has travelled a distance $D2$ from a mirror image of the gun, and arrives $(D2—D1)/c_\infty$ later, as shown in Fig. 11.

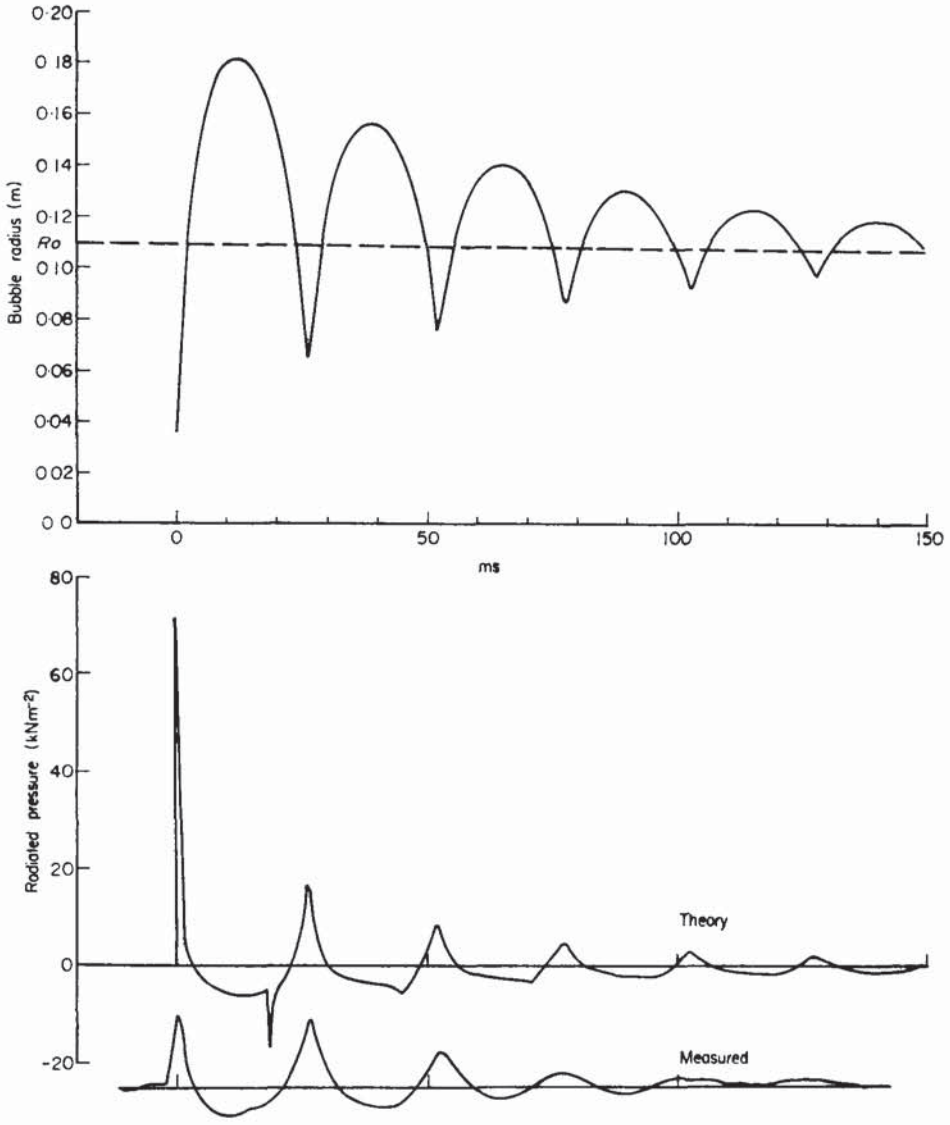


FIG. 9. The same as Fig. 7 with artificial damping included. A damping constant $\alpha = 11{\cdot}0\ s^{-1}$ has been used.

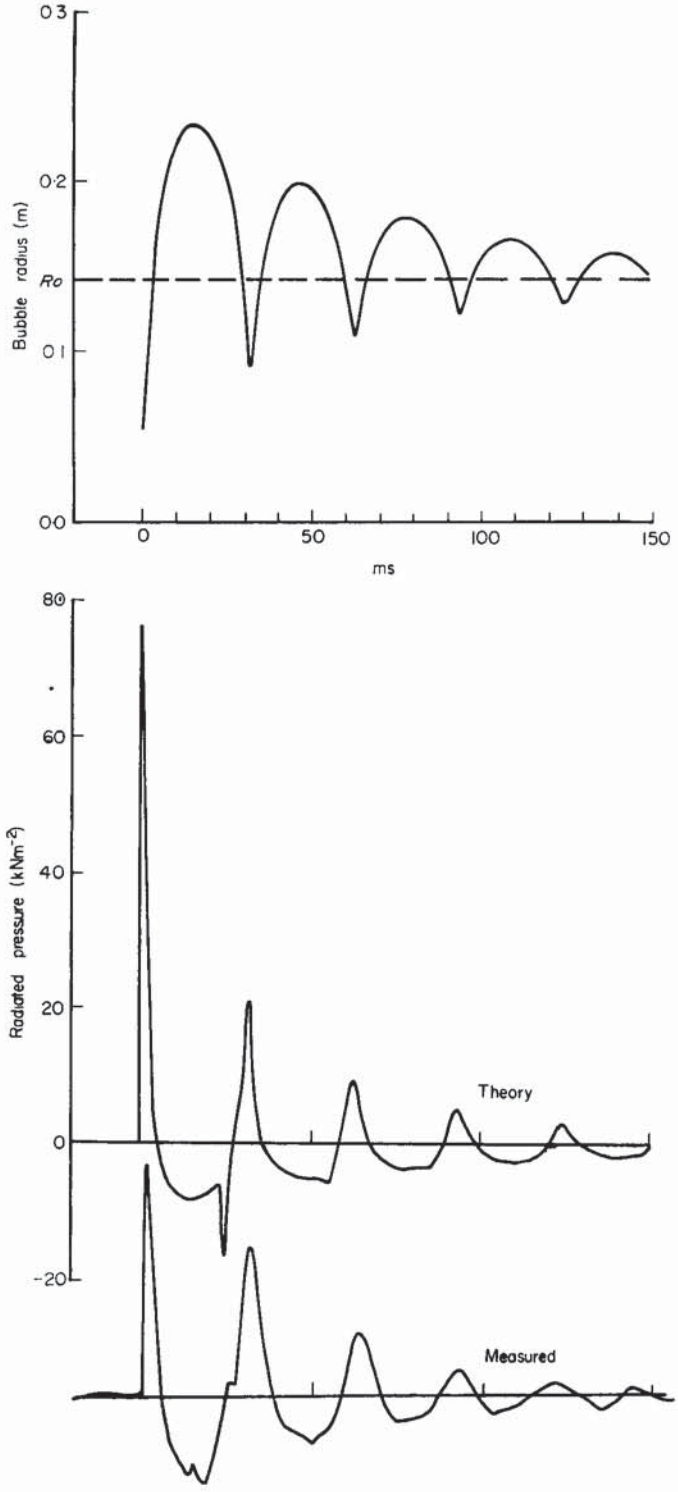


FIG. 10. The same as Fig. 8 with artificial damping included. A damping constant $\alpha = 11{\cdot}0\ s^{-1}$ has been used.

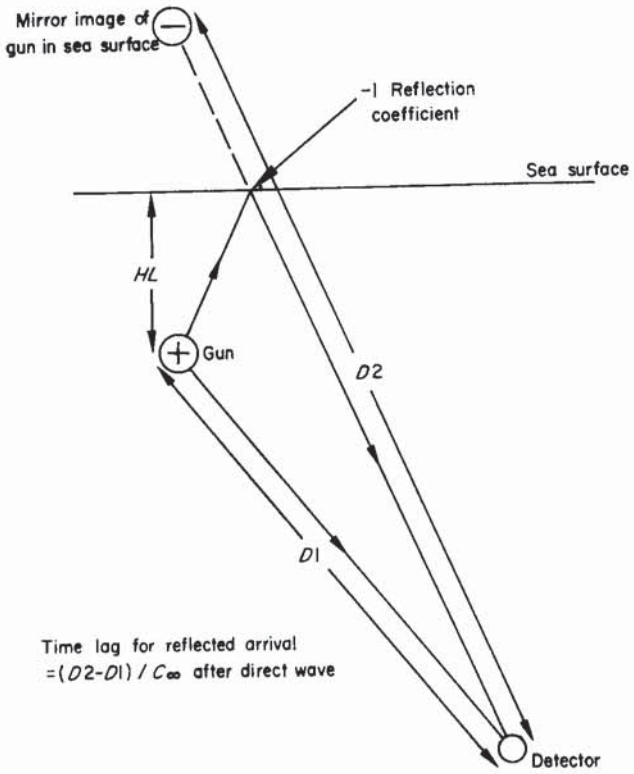


FIG. 11. Diagram to show superposition of direct and reflected pressure waves at a detector.

Since an unrealistic exponent b in equation (47) and the addition of damping equations (57)–(60) were found to be necessary to produce the correct sort of radiation from the bubble model, it might be surmized that the same result could have been obtained using only a first-order theory. However, since the second-order theory is really necessary to determine the acoustic radiation from these large amplitude oscillations at distances far from the bubble, we must have the bubble-wall motion, the boundary condition for these equations, accurate to the same order.

4. Discussion of waveforms as predicted using the above theory and some applications

Discussion

It can be seen from Figs 9 and 10 that the general shape of the theoretical waveforms is a reasonably close approximation to the experimental ones. The periodicity is about the same, showing that it is reasonable to assume a constant value of b throughout the motion, and to keep its value the same for both guns. It is observed that the period of the radiated pressure wave decreases (after the first one or two oscillations) slightly faster than using the theory and assuming the centre of the bubble does not move. Herring (1949) has shown that this effect is due to the proximity of a free surface, the distance between the bubble and the surface being small compared to the wavelength. His argument is as follows:

'The water between the bubble and the surface can be more easily given a radial acceleration when the surface is near than when it is distant. Consequently the streamlines tend to bend towards the surface, with the result that, for a given a and

da/dt (where a is the radius of the bubble), the kinetic energy of the water is less than the value $2\pi\rho a^3(da/dt)^2$ which applies when there is no free surface. Thus the effect of the surface is like decreasing the inertia of a simple oscillator without changing the spring constant, and so it decreases the period.'

This effect is not particularly marked at normal operating depths.

It can be seen that the inclusion of the damping equations has had the effect of reducing the sharpness of the pressure peaks as well as to produce the right sort of decay. The damping variable, α, has been kept constant throughout the motion. Whether α is something which will vary from gun to gun, depending on how neatly the gun can put all its air into a spherical bubble, is hard to say with the limited data available. However, it can be seen that a constant value for α of 11·0 has the desired effect on the predicted waveforms.

The one worrying aspect of the theoretical waveforms is the inevitable appearance of the very sharp initial peak. This appears again when the reflected wave is included. This is purely a function of the initial conditions discussed in Section 3. If an air gun were capable of opening infinitely fast, causing all of its air to be transferred into one bubble, this peak would be seen. However, the measured peaks are less as the real conditions are more complicated and vary from gun to gun.

Since these peaks are just a function of starting conditions, only the first values in the digital direct and reflected waveforms as calculated by the above method will be spuriously high. This does not lead to embarrassment when one comes to use the theoretical waveforms to design digital filters for processing, as will be shown elsewhere.

Applications

When using air guns in seismic reflection profiling at sea, provided the parameters HL, V, X are known at the time of firing, the pressure waveform at any point in the sea can be calculated, as above, to sufficient accuracy for some use to be made of its properties. As long as bubbles from individual guns are independent—at least three bubble diameters apart—the resultant waveform can be predicted by superposing the individual waveforms.

Power spectra of the calculated waveforms may be used to design more efficient systems; and because all the reflections of the waves propagated by various bubbles can be isolated, the spectra of pressure waves which cannot be measured—in particular those about to enter the sea-floor immediately below the survey ship—can easily be found with this theory.

An equally useful application of this theory is in the digital processing of the returning waves from reflectors beneath the sea-floor. (In preparation.)

Acknowledgments

I am very grateful to Mrs S. L. Jupp and Dr D. P. McKenzie for many helpful discussions. I am also indebted to Dr A. P. Stacey and Dr F. Gray for their encouragement.

Dr F. Gray, Mr M. Mason, Mr T. R. E. Owen, Mr R. A. Scrutton and Dr A. P. Stacey all helped in the measurement of the bubble-pulse waveforms, while I was semi-incapacitated, having just broken my foot.

The smaller air gun was lent to us by Mr H. Peacock of Geophysics Department, University of Durham, for which I thank him.

Department of Geodesy and Geophysics,
Cambridge University.

References

Ewing, J. I. & Zaunere, R., 1964. Seismic profiling with a pneumatic sound source, *J. geophys. Res.*, **69**, 4913–4915.

Flynn, H. G., 1964. Physics of acoustic cavitation in liquids. In *Physical Acoustics*, Vol. 1B (W. E. Mason, ed.), 57–172, Academic Press.

Giles, B. F., 1968. Pneumatic acoustic energy source, *Geophys. Prospect.*, XVI, No. 1, 21–53.

Gilmore, F. R., 1952. Collapse of a spherical bubble. Report No. 26–4, Hydrodynamics Laboratory, California Institute of Technology, Pasadena, California.

Kaye, G. W. C. & Laby, T. H., 1957. *Tables of Physical and Chemical Constants*, Longmans, London.

Kirkwood, J. G. & Bethe, H. A., 1942. OSRD Report No. 588.

Herring, C., 1949. Theory of the pulsations of the gas bubble produced by an underwater explosion. In *Underwater Explosion Research*, Vol. 2, 1950. Office of Naval Research, Dept. of the Navy.

Landau, L. D. & Lifshitz, E. M., 1959. *Course of Theoretical Physics*, Vol. 6, Pergamon Press Ltd., Oxford.

Taylor, G. I., 1942. Vertical motion of a spherical bubble and the pressure surrounding it. In *Underwater Explosion Research*, Vol. 2, 1950. Office of Naval Research, Dept. of the Navy.

Appendix I

Neglecting gravity

The gravity problem is a very complicated one. Because of the existence of gravity the pressure distribution around the bubble cannot be uniform and therefore the bubble cannot be spherical.

G. I. Taylor (1942) has shown that the displacement due to the buoyancy effect of the bubble in the gravity field may be treated as if the bubble were constrained to be spherical and deduces the rise times and rates in close agreement with experiment. He shows that much of the vertical momentum of the bubble is acquired when the bubble is large, and produces consequently large velocities when the bubble contracts.

One would expect that the oscillating bubble might then behave as if its centre were nearer and nearer the surface, and so oscillate with a longer and longer period. But because of the free-surface effect already described (Section 4), the period of oscillation should, on the contrary, get shorter as the bubble rises, as observed.

In conclusion one can say that it is convenient to neglect gravity because we get reasonable answers without considering it, and to take the effect into account is too difficult.

Appendix II

Errors in the measurement of the period of oscillation

The theory shows that the period of oscillation depends on the input parameters: volume, V; depth of gun, HL; gauge pressure at instant of firing, X; atmospheric pressure, po. Of these, po can be considered to be constant at one standard atmosphere, since the variations experienced about this value lead to no significant variations in the period of oscillation.

The volume of the gun can easily be measured to sufficient accuracy for no significant error in the period to be incurred. However, it is not known how much of the air in the gun can escape to the bubble, therefore the error in V is uncertain.

The other parameters, HL and X, were measured using pressure gauges. The estimated error in reading the gauge to measure X is ± 20 p.s.i.g. Judging from the behaviour of the depth sensor, the depth could only be estimated to ± 2 feet. Plots of the variation of period with depth and period with pressure for data from the 0·5 l gun are shown in Figs 12 and 13. The error in pressure measurement can result in an error in the calculated period of $\pm 0{\cdot}12$ ms. The error in the depth measurement can result in an error of $\pm 0{\cdot}55$ ms.

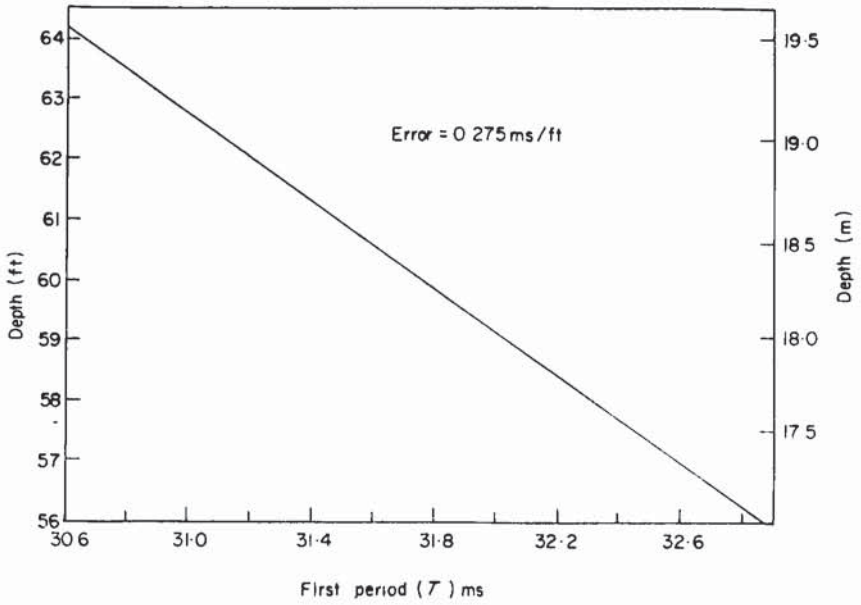


FIG. 12. Graph to show variation of predicted first period of oscillation (T) with depth (HL) for the following data: $V = 0{\cdot}5$ l; $X = 10{\cdot}6$ MNm^{-2}.

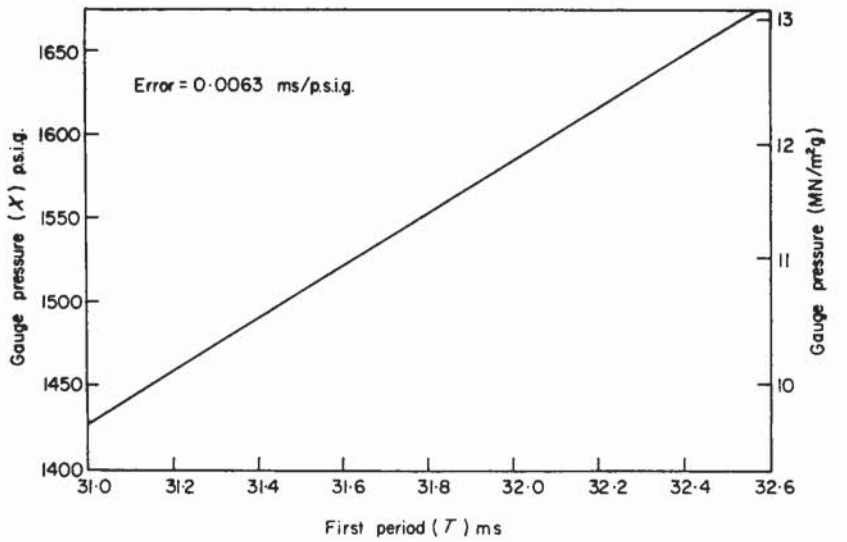


FIG. 13. Graph to show variation of predicted first period of oscillation (T) with gauge pressure (X), for the following data: $V = 0{\cdot}5$ l, $HL = 18{\cdot}3$ m.

There is also a possible error in measuring the period observed on the ultra-violet paper recordings, estimated to be $\pm 0{\cdot}25$ ms from the thickness of the lines produced by the moving light spot. Hence, under the assumption that all the air from the gun enters the bubble, that is, neglecting the unknown error in V, the possible experimental error in estimating the period of oscillation is $\pm 0{\cdot}12 \pm 0{\cdot}55 \pm 0{\cdot}25 = \pm 0{\cdot}92$ ms. These errors will be about the same for all readings from both guns.