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The signature of an air gun array: Computation from near-field measurements including interactions

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ABSTRACT

We designed a system to enable the signature of an air gun array to be calculated at any point in the water from a number of simultaneous independent measurements of the near-field pressure field [subject of a patent application]. The number of these measurements must not be less than the number of guns in the array.

The underlying assumption in our method is that the oscillating bubble produced by an air gun is small compared with the wavelengths of seismic interest. Each bubble thus behaves as a point source, both in the generation of seismic waves and in its response to incident seismic radiation produced by other nearby bubbles. It follows that the interaction effects between the bubbles may be described in terms of spherical waves.

The array of interacting guns is equivalent to a notional array of noninteracting guns whose combined seismic radiation is identical. The seismic signatures of the equivalent independent elements of this notional array can be determined from the near-field measurements. The seismic radiation pattern emitted by the whole array can be computed from these signatures by linear superposition, with a spherical correction applied.

The method is tested by comparing far-field signatures computed in this way with field measurements made in deep water. The computed and measured signatures match each other very closely. By comparison, signatures computed *neglecting* this interaction are a poor match to the measurements.

INTRODUCTION

The signature of an air gun array is inconveniently long and oscillatory. Its spectrum is multipeaked and is not minimum phase. Yet, despite these obvious disadvantages, the air gun has become the most widely employed marine seismic source because of its renowned reliability and signature repeatability.

To overcome the undesirable aspects of the signature of a single gun, arrays of guns of different sizes are normally used together to create a composite source whose signature characteristics are more desirable. The usual design aim for such an array is that it should generate a seismic wave whose signature in the vertical-downward direction is short and sharp (that is, with a large primary-to-bubble ratio) and whose spectrum is smooth and broad over the frequency band of interest (and preferably minimum phase) (Giles and Johnston, 1973; Nooteboom, 1978; Brandsaeter et al, 1979).

Although much progress has been made in trying to meet this aim, there is a chronic problem in determining the signature.

Once the number of guns in an array is sufficient to create a signature whose spectrum is adequately broad and smooth, the dimensions of the array are not small compared with the wavelengths of sound generated. This has two consequences for the seismic radiation pattern of the array. First, it forces the signature to vary with direction. Second, it causes the signature of the pressure wave to vary with distance; that is, in a given direction, the phase spectrum of the pressure wave is distance-dependent. It becomes independent of distance only in the "far field" of the array at distances greater than about D^2/λ, where D is the dimension of the array and λ is the wavelength of interest.

Far-field measurements of the seismic radiation of an air gun array cannot normally be made on the continental shelf, because there is insufficient depth of water to prevent the measurements from being severely contaminated by sea-bottom reflections. The measurements must be made in deep water, and it is extremely difficult to determine the true relative positions of the array and measuring device with any precision. The measurement of the far-field radiation pattern is thus fraught with difficulties, and it makes sense to try to calculate this signature from near-field measurements.

If interactions between the air guns within an array were negligible, it would be possible to superpose the signatures of individual guns to calculate the far-field signature of the array. Since the dimensions of an air gun bubble are very small compared with a wavelength, the distance at which the phase spectrum of the

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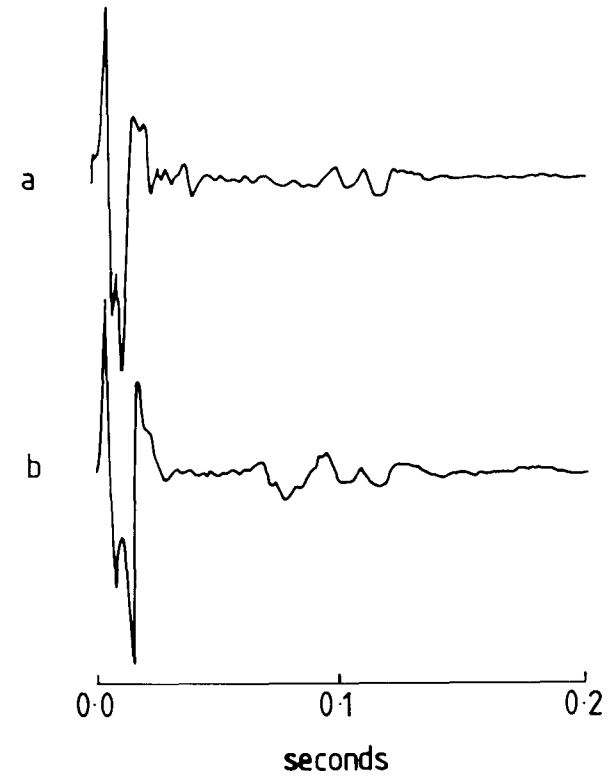


FIG. 1. Comparison of (a) signature measured 150 m below a seven-gun array with (b) signature computed by superposing individual gun signatures. Filter setting: out 360 Hz.

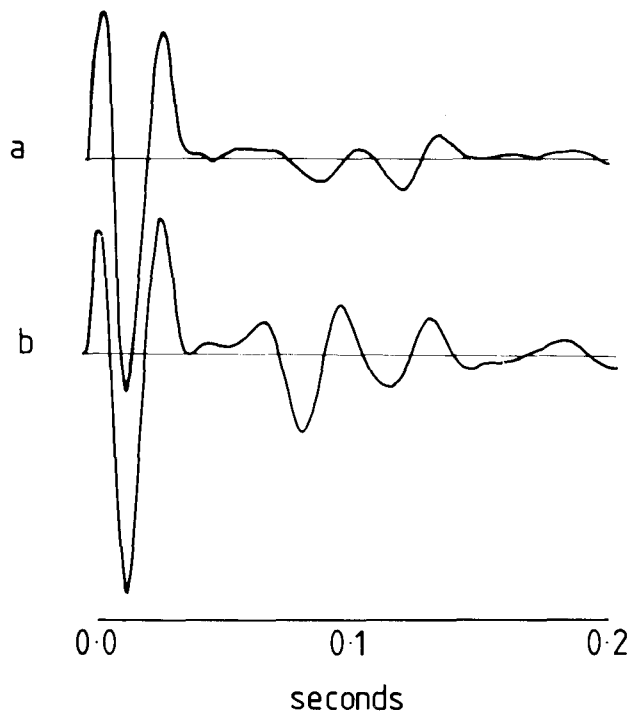


FIG. 2. Same signatures as in Figure 1, but with 40 Hz high-cut filter applied. (a) is measured signature, (b) is computed by simple superposition.

radiated pressure wave becomes independent of range from the gun is very small, and measurement of this wave can be made very close to the gun. This approach has obvious attractions.

However, interactions between the air guns are not negligible, especially at low frequencies. Superposition does not apply, and this approach fails to yield the correct signature.

We offer a solution to this problem which takes the interactions into account. It requires measurements of the pressure field near each gun in the array, from which the signature may be calculated in any direction and at any distance. In particular, we may determine the far-field radiation pattern of the array, as required.

INTERACTION, SUPERPOSITION AND THE LAW OF CONSERVATION OF ENERGY

We may superpose the wave fields of a number of sources, each acting independently, to obtain the resultant wave field of all the sources acting together, provided none of the sources is affected by the wave field produced by any of the others. That is, superposition applies if interaction between sources is negligible. Interaction is simply the wave field of one source affecting the way another source behaves, and vice versa.

Figure 1a is a measured far-field signature of a seven-gun array in which the hydrophone is placed 150 m below the array in deep water. Figure 1b is a calculated far-field signature equal to the superposition of individual far-field measurements of the seven guns in the array. The two signatures are not identical; therefore, superposition does not apply. There are obvious errors in the amplitude of the calculated signature; there are also important errors in phase which are not very clearly recognized in this broadband display.

Figures 2a and 2b show the same two signatures filtered with a 40 Hz low-pass filter. The amplitude discrepancy is still as apparent, but the phase distortion is now more noticeable—the peak-to-bubble period of the calculated signature is clearly shorter than that of the measured one.

Comparing Figures 1 and 2, we see that phase differences between the calculated and measured signatures are more discernible at low frequencies. This indicates that the superposition assumption breaks down at low frequencies. Or, conversely, interaction between air guns within the array is more important at low frequencies than at high frequencies.

We may demonstrate the accuracy of this deduction by considering the implications for the law of conservation of energy which arise when interaction between sources is ignored.

Consider two point sources a distance D apart, and consider a point Q in the far field a distance r from both sources as shown in Figure 3. Let the pressure wave at Q radiated by the first source, acting independently, be $p_1(t)$, and let the corresponding wave from the second source be $p_2(t)$. Since Q is in the far field, the particle velocity wave is in phase with the pressure wave (Ziolkowski, 1980, Appendix 2) by definition, and we have

$$u_1(t) = \frac{p_i(t)}{\rho c}, \quad i = 1 \quad \text{or} \quad 2, \tag{1}$$

where ρ is the density of the water, and c is the speed of sound. The energy radiated by either source on its own is

$$\begin{aligned} E_i &= 4\pi r^2 \int_0^\infty u_i(t) \cdot p_i(t)\, dt \\ &= \frac{4\pi r^2}{\rho c} \int_0^\infty p_i^2(t)\, dt, \quad i = 1 \quad \text{or} \quad 2. \end{aligned} \tag{2}$$

(1) Assume there is no interaction between the sources

In this case each source is unaffected by the other, and the radiated energy of the two sources together is equal to the sum of the two acting separately:

$$E = E_1 + E_2 = \frac{4\pi r^2}{\rho c} \sum_{i=1}^{2} \left[\int_0^\infty p_i^2(t)\,dt \right]. \quad (3)$$

We may apply the law of superposition to obtain the resultant wave incident at Q, which is

$$p_Q(t) = p_1(t) + p_2(t) = \sum_{i=1}^{2} p_i(t). \quad (4)$$

(2) Assume D is small compared with a wavelength λ

Then the two sources taken together are a point source which has no directivity pattern. The radiation has spherical symmetry at the frequencies of interest. We may again calculate the energy as:

$$E = \frac{4\pi r^2}{\rho c} \int_0^\infty p_Q^2(t)\,dt = \frac{4\pi r^2}{\rho c} \int_0^\infty \left[\sum_{i=1}^{2} p_i(t) \right]^2 dt. \quad (5)$$

Now the two expressions for the energy [equations (3) and (5)] are not the same. If for example $p_1(t) = p_2(t)$, equation (3) would yield $E = 2E_1$, which is consistent with the law of conservation of energy, while equation (5) would yield $E = 4E_1$, which is not. It follows that if the two sources are far enough apart that interactions between them may be neglected, they must also be far enough apart to have a directivity pattern. The *constructive* interference at Q is compensated by *destructive* interference in other directions, such that the law of conservation of energy is obeyed. In order for this interference pattern to exist, the distance D must be of the order of a wavelength or larger. If D is less than λ, we cannot assume superposition applies, because we would then violate the first law of thermodynamics, as we have shown above.

We may therefore make the following observation. Interaction between sources is significant if the ratio of the distance between them to the wavelength, D/λ, is less than 1. Since $\lambda = c/f$, it follows that interaction is more important at low frequencies than at high frequencies, as observed.

THE THEORY OF INTERACTION BETWEEN BUBBLES

Previous work

The problem of interaction between the oscillating bubbles of an air gun has been examined many times, and the conclusions are always different. Ziolkowski (1970 p. 158), for example, said that it is not even a problem: "As long as bubbles from individual guns are independent—at least three bubble diameters apart—the resultant waveform can be predicted by superposing the individual waveforms." This remark was made out of sheer ignorance and is retracted here. The maximum bubble diameter is small compared with the shortest wavelength of interest (Ziolkowski, 1970); three bubble diameters is still *fairly* small compared with the shortest wavelength of interest, so interaction between bubbles with that separation will be significant at all frequencies.

Giles and Johnston (1973) produced some not very clear evidence to reach roughly the same conclusion as Ziolkowski (1970). Their criterion for minimum gun separation to avoid significant interaction was accepted by Nooteboom (1978) who gave it a formula. It could be argued that everyone means something slightly different by the words "significant interaction." However, it seems what we have shown above is that no definition could be complete without including a reference to the frequency range of interest, and it is exactly that which is missing from the discussions of interaction in the three papers cited immediately above.

Lugg (1979) did some very careful work to show that interaction between two identical 120 inches^3 (1.97 ℓ) guns is just noticeable when the guns are 480 inches (12.2 m) apart. The results shown by Lugg in his Figure 62 are measurements made with a wide filter setting which include the higher frequencies at which we would expect negligible interaction at such distances. The importance of the interaction would be even more noticeable if these high frequencies were filtered out.

Safar (1976) discussed the interaction that can occur between identical bubbles which have small oscillations about an equilibrium value. Such bubbles, which do not closely resemble the bubbles produced by air guns (Ziolkowski, 1977), can be modeled as damped linear oscillators. The interaction can then be treated as a change in the radiation impedance load seen by each oscillator. One key feature of Safar's analysis is that the interaction is treated

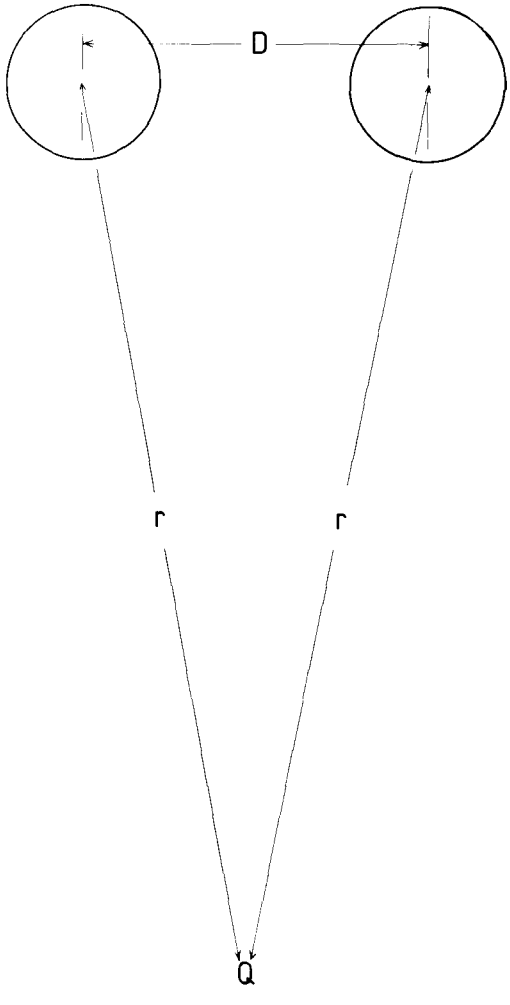


FIG. 3. The sound wave generated by two point sources a distance D apart, as seen at a point Q in the far field a distance r from each source.

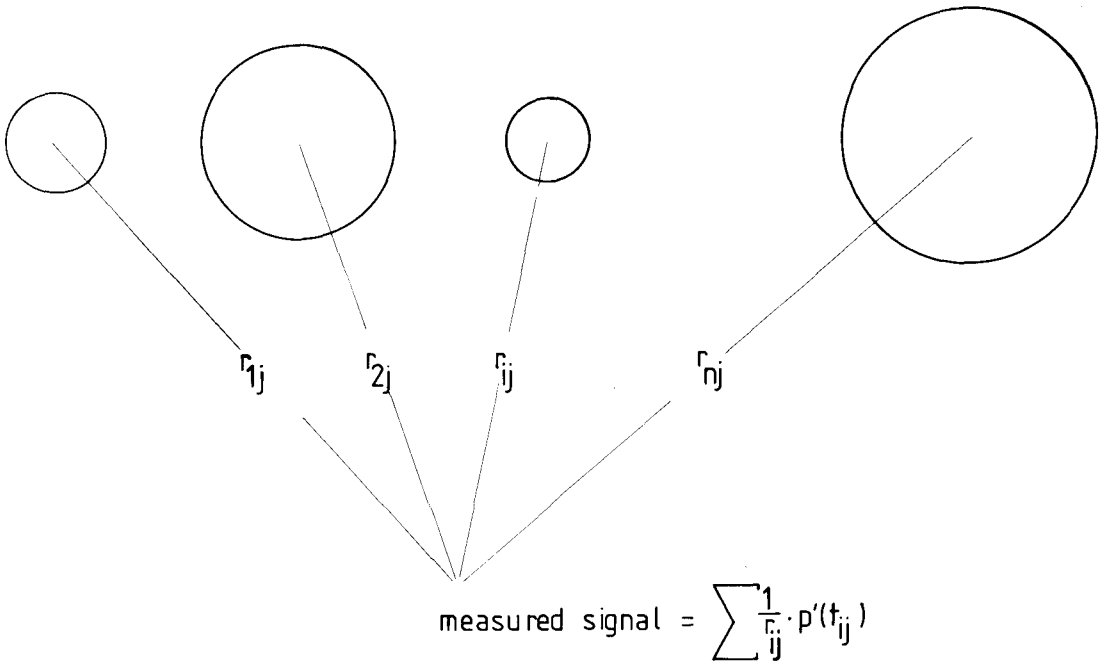


FIG. 4. The signature seen at a hydrophone in the vicinity of an air gun array. The array of interacting bubbles behaves as if it were an equivalent array of independent, noninteracting notional bubbles in the same locations.

as a modulation of hydrostatic pressure, which is exactly what we do here.

A more general approach to the interaction problem was presented by Sinclair and Bhattacharya (1980) who dealt with nonimpulsive sources. Their analysis clearly indicated that interaction is a frequency-dependent phenomenon and that it is the modulation of the hydrostatic pressure field which is responsible. They were not sure how to proceed with impulsive sources but concluded that "it is probable that the linear wave theory would be adequate to describe the coupling between sources, but that the non-linearity of the radiator itself (e.g., the air bubble) would have to be taken into account" (p. 331). We agree.

Our approach

Consider first a single air gun, consisting of a chamber of air at high pressure which is suddenly opened. The escaping air forms a bubble which expands very rapidly against the water. As it expands, the pressure in the bubble drops, and even drops to below the hydrostatic pressure of the water, because the inertia of the moving water carries the expansion through this equilibrium position. The expansion then begins to slow down (because the pressure differential is now acting inward) and then finally stops. The bubble then collapses, overshooting the equilibrium position again while the internal pressure increases. The collapse of the bubble is halted by the rapid internal pressure build-up; at this point the oscillation is ready to begin again.

The oscillating bubble is a seismic wave generator. Because the bubble diameter is always small compared with the seismic wavelengths, this wave has spherical symmetry at seismic frequencies. As shown in the Appendix, the phase spectrum of the pressure wave generated by the bubble is the same at all distances at which the linear elasticity theory (Hooke's law) applies. The amplitude of the wave is inversely proportional to the distance. Thus, at a distance r the transmitted wave would be

$$\frac{1}{r} \cdot p\left(t - \frac{r-1}{c}\right),$$

where the time origin has been chosen as if the wave had originated at a point 1 m from the center of the bubble, and r is in meters. The so-called "afterflow term" (Keller and Kolodner, 1956; Kramer et al, 1968) is absolutely negligible in the range where the linear wave theory applies (see the Appendix).

The driving mechanism behind this oscillation is the pressure difference $P_d(t)$ between the inside of the bubble $P(t)$ and the hydrostatic pressure P_H. The hydrostatic pressure remains virtually constant throughout the oscillation because the movement of the buoyant bubble towards the surface is very slow. Thus

$$P_d(t) = P(t) - P_H. \tag{6}$$

If $P_d(t)$ is positive, it tends to make the bubble expand, or slow down the collapse. If $P_d(t)$ is negative, it tends to make the bubble collapse, or slow down the expansion.

Now let us consider n guns. If they were fired independently, the driving pressure at the ith gun would be

$$P_{di}(t) = P_i(t) - P_{Hi}. \tag{7}$$

When the guns are fired together, this behavior is modified. In particular, the pressure around each bubble is no longer constant. Sound waves from many directions impinge on each bubble, modifying its behavior.

These sound waves are the radiation from the other bubbles, and they contain energy at seismic frequencies. At the ith bubble

there will be pressure differences between one side of the bubble and the other, but at seismic frequencies these pressure differences will not be discernible, because the bubble is small compared with the seismic wavelengths. At these wavelengths the bubble appears to be a point. The effect of this radiation field on the bubble is simply to add a time-varying component to hydrostatic pressure. The water pressure at the ith bubble thus becomes time-variant:

$$P_{Hi}(t) = P_{Hi} + m_i(t), \tag{8}$$

where $m_i(t)$ is the modulating pressure field at the ith bubble. Thus the pressure in the water is the sum of *hydrostatic* pressure and the *dynamic* term $m_i(t)$.

The driving pressure in the bubble of the ith gun is the difference between the internal pressure and the pressure in the water

$$P'_{di}(t) = P'_i(t) - P_{Hi}(t), \tag{9}$$

where primes indicate the change in behavior due to interaction. The dynamics of the ith bubble are affected by the changes in water pressure. Therefore the rates of expansion and collapse of the bubble are different. The internal pressure $P_i(t)$, in the absence of any influence from other bubbles, is different from the internal pressure $P'_i(t)$ when this influence is taken into account. It follows that the seismic wave $p'_i(t)$, generated by the ith bubble under the influence of the other bubbles, will be different from the wave $p_i(t)$ generated when the bubble is independent. Combining equations (8) and (9), we have

$$P'_{di}(t) = [P'_i(t) - m_i(t)] - P_{Hi}. \tag{10}$$

Comparing equations (7) and (10), we see that the modified

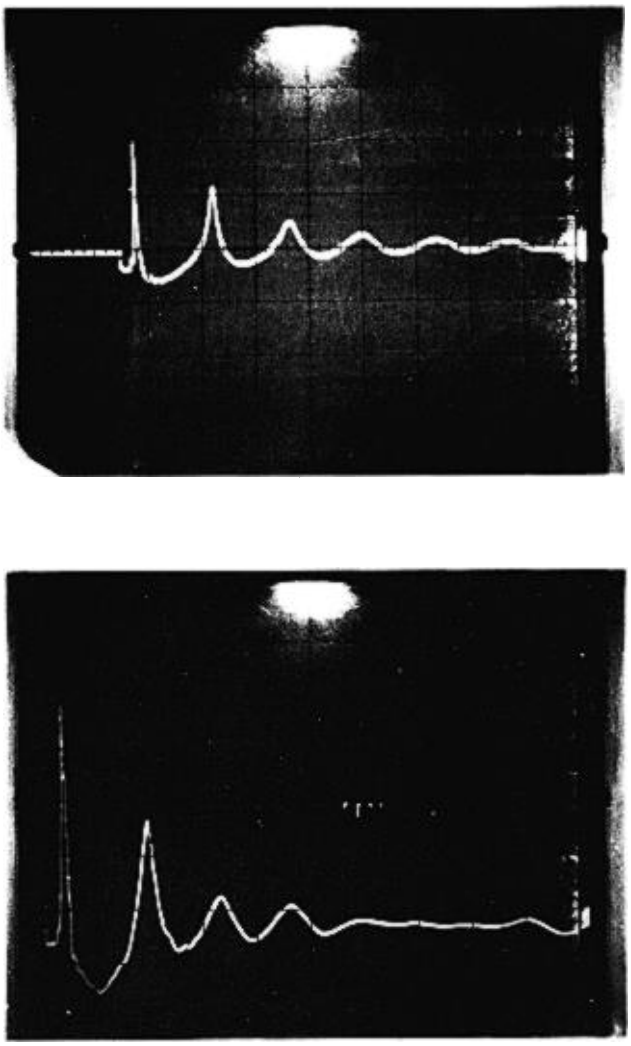


FIG. 5. (a) The signature 1 m away from a 95 inches^3 air gun at 2000 psig firing pressure. (b) The signature registered by the same hydrophone in the same location when the full array is firing. The time scale is 50 msec per division. The amplitude scales are not necessarily the same.

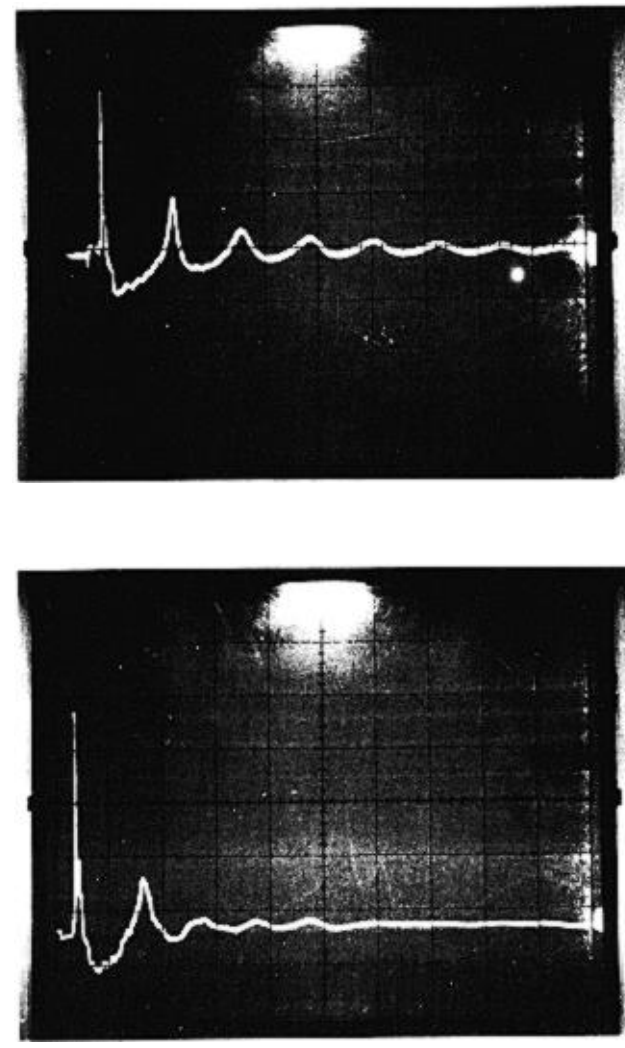


FIG. 6. (a) The signature 1 m away from a 50 inches^3 air gun at 2000 psig firing pressure. (b) The signature registered by the same hydrophone in the same location when the full array is firing. The time scale is 50 msec per division. The amplitude scales are not necessarily the same.

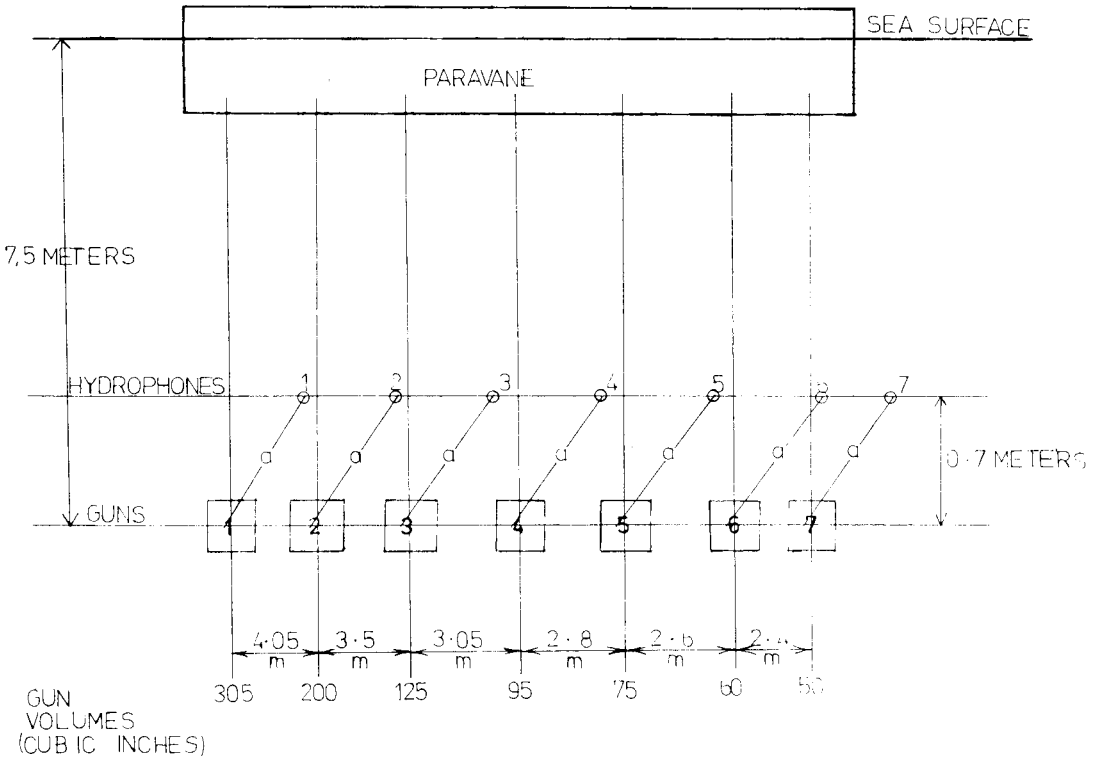


FIG. 7. Geometry of the array, showing the positions of the near-field hydrophones. The distance $a = 1$ m.

bubble behaves as if it were an independent bubble, oscillating in water with constant pressure P_{Hi}, with its internal pressure varying as

$$P_i'(t) - m_i(t),$$

with the net result that its signature becomes $p_i'(t)$.

We do not know what $P_i(t)$ or $p_i'(t)$ are, of course. These are merely names given to modified internal pressure and radiated signature. The point is that we have described the interacting bubbles in such a way that they are now equivalent to independent "notional" bubbles with modified signatures $p_i'(t)$. Since the bubbles do not change greatly in size under the influence of this interaction, they are all still small compared with the wavelengths of seismic radiation they radiate, and their radiation has spherical symmetry. If we place hydrophones in the radiation field of these bubbles, we can write the pressure field that they will see as follows:

$$p_j(t) = \sum_{i=1}^{n} \frac{1}{r_{ij}} p_i'(t_{ij}), \tag{11}$$

where

$$t_{ij} = t - \frac{r_{ij} - 1}{c}, \tag{12}$$

and r_{ij} is the distance from the ith bubble to the jth hydrophone, as shown in Figure 4.

If we know the geometry, that is the distances r_{ij}, we can see from equation (11) that there are n weighted unknown signatures $p_i'(t)$ which are combined, with phase delays, to yield the measurable signature $p_j(t)$ at the jth hydrophone. If we have n such hydrophones, we may make n independent measurements to solve for the n unknowns

$$p_j(t) = \sum_{i=1}^{n} \frac{1}{r_{ij}} p_i'(t_{ij}), \quad j = 1, 2, \ldots n. \tag{13}$$

In summary, we can solve the problem of interaction between oscillating bubbles, provided we make n independent measurements of the pressure wave field surrounding these bubbles and provided we place our hydrophones in the linear radiation field.

SIGNATURE PREDICTION FROM NEAR-FIELD MEASUREMENTS

We need n independent hydrophones, each of known sensitivity such that the received voltage at each hydrophone is

$$h_j(t) = s_j, p_j(t). \tag{14}$$

From the form of equation (13) it is clear that we can make life simple for ourselves if we put the hydrophones in the right places, that is, if we choose the r_{ij} carefully. In particular, if we try to put one hydrophone very close to each gun, then the biggest term in the n-term summation at any hydrophone will be the wave generated by the nearest gun since the amplitude varies inversely proportional to the distance from the gun. We can obviously exploit this information in our solution of equation (13).

Figures 5 and 6 show the effect of interaction on the near-field

measurement of two guns in a subarray. In both cases the upper signature is an unfiltered measurement of the pressure signature $P_j(t)$ at 1 m from the gun, fired on its own; the lower signature is measured in the same place with all guns fired together. (The amplitude scale is not the same on all signatures.) The lower signature is the weighted sum of all the modified signatures $P_i'(t)$, as given by equation (13). However, we note that as a result of our geometry (see Figure 7), the biggest contribution to the measured signature is the weighted signature from the nearest gun.

Equations (13) may be solved iteratively (with such geometry), or by some linear inverse method, to obtain the n modified independent signatures $P_i'(t)$, given the n measurements. Once the signatures are known, equation (13) may be used in reverse, that is, to predict the signature at any point in the water.

Figure 8 shows a comparison of a measured far-field signature of an air gun array with calculated signatures. Figure 8a is the prediction made from near-field measurements, described above, including interaction and assuming a sea surface reflection coefficient of -1.0; Figure 8b is the measured signature; Figure 8c is the prediction made from far-field measurements of the guns fired independently, with interaction ignored, that is, assuming linear superposition applies. Clearly, linear superposition does not apply. The computation based on the interaction theory described above gives a result very close to the measurement. Figure 9 shows the same three signatures as in Figure 8, but they have all been filtered with a 40 Hz high-cut filter. Agreement between measurement (b) and the calculation including interaction (a) is very close. The agreement between measurement (b) and the calculation neglecting interaction (c) is very poor. This reinforces our contention that it is very important to include the interactions.

There are two problems which can arise in making these predictions. One concerns the relative velocity of the hydrophones and the bubbles in operation. The other concerns the wave propagation through the water containing bubbles.

The first problem is simple to understand, but it may not be so easy to solve. The gun array is towed through the water at about 5 knots in normal operation. When the guns are fired, the bubbles oscillate in water through which the guns, gun harness, and hydrophones are towed. There will be some drag in the direction

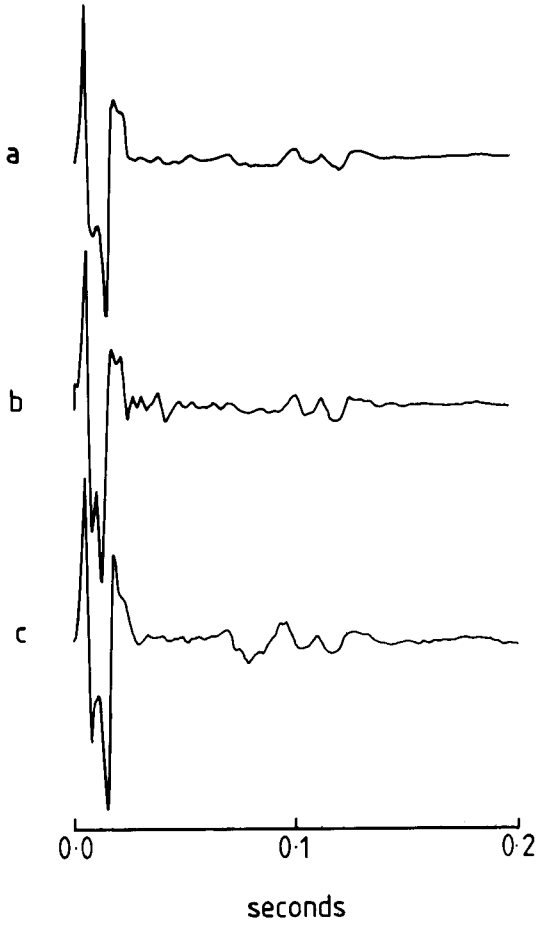


FIG. 8. Comparison of (a) the far-field signature computed from near-field measurements taking the interactions into account, with (b) the signature measured 150 m below the full seven-gun array, and (c) the signature computed by simple superposition of the seven independent signatures, neglecting interaction.

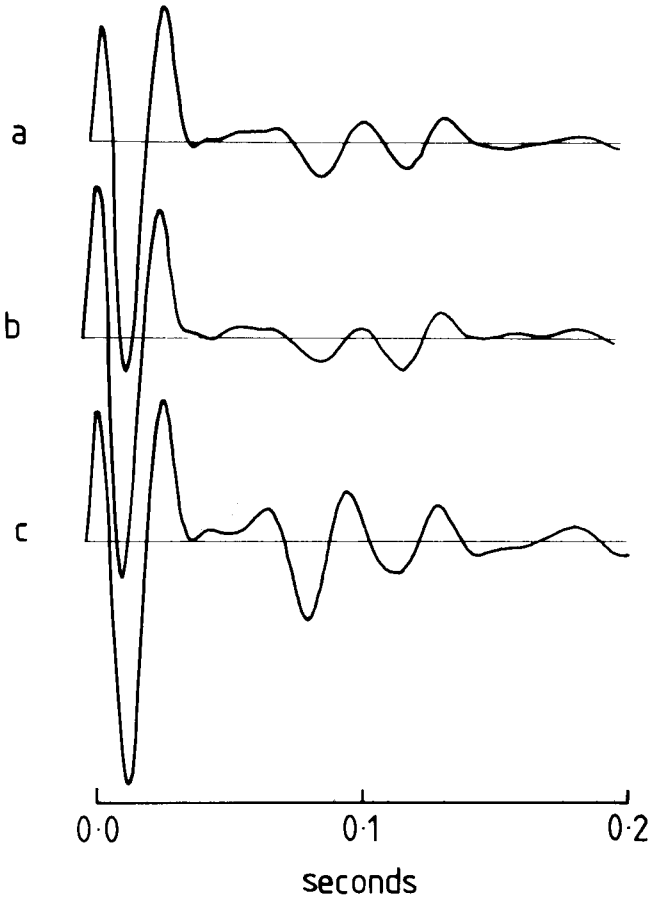


FIG. 9. Same signatures as in Figure 8, but with 40 Hz low-pass filter applied. (a) Computed including interaction; (b) measured; (c) computed neglecting interaction.

of the boat, but the bubbles will tend to be left behind. Their buoyancy will also tend to make them rise. Thus, between the ith bubble and the jth hydrophone there will be a closing relative velocity at time t of $v_{ij}(t)$. The distance between the ith bubble and the jth hydrophone becomes

$$r_{ij} - t \cdot v_{ij}(t),$$

such that the modified equations (13) become

$$p_j(t) = \sum_{i=1}^{n} \left(\frac{1}{r_{ij} - tv_{ij}} \right) p_i'(t_{ij}),\ j = 1, 2, \ldots n. \quad (15)$$

In our computations we have ignored the velocity term, but it is clear from equation (15) that its importance is greatest when the hydrophone-bubble separation r_{ij} is least.

The second problem concerns the inhomogeneity of the water. The wave field generated by each gun must propagate through the water. But this water contains bubbles. Domenico (1981) showed that the influence of bubbles on the attenuation and velocity of propagating sound waves can be enormous. We have neglected such effects here. Are we justified in doing so? We believe we are.

In his experiment, Domenico created air curtains in a pool of water through which the sound waves had to propagate. Our situation is different. There is an enormous body of water through which the sound wave may propagate, and because there are only a few bubbles, each of which is small compared with a wavelength, their influence on the waves can be only slight. We believe we may ignore the effect the bubbles have on the overall compressibility of the water. Our experimental results confirm this.

CONCLUSIONS

We present a theory of interactions between the bubbles produced by an air gun array which shows that the array is equivalent to a notional array of noninteracting oscillating bubbles. Each of these notional bubbles emits a propagating spherical pressure wave whose amplitude decreases inversely with the distance and whose phase spectrum is invariant with distance. If there are n guns in the array, we need n independent measurements of the pressure field in order to determine the n notional source signatures. The measurements are best made in the near field.

Once these n signatures have been determined from the measurements, it is a simple matter to determine the pressure signature at any point in the water. In particular, we may compute the far-field signature of the array in any direction. This is confirmed by experiment. The interaction theory therefore allows us to compute the full directivity response of the array from near-field measurements made in operation. Our method has thus made the conventional deep-water measurement of "the far field signature" redundant.

In summary, we present a method which is a deterministic solution to the problem of finding the far-field signature of an air gun array in any direction. All it requires is an adequate number of measurements of the near-field pressure field. Because it is deterministic, our method places no special constraints on the required phase spectrum of the outgoing wave. For example, there is now no need to throw away useful signal energy in order to ensure that the downward-traveling wave has a large primary-to-bubble ratio. This so-called desirable characteristic has been an industry requirement only because it has been necessary to resort to statistical techniques to find and extract the signature from the recorded data. Now we have a deterministic method, and we are free to design the array to generate a signal with the power spectral characteristics we desire, without regard to the phase. The issues involved in the design of such an array are, however, beyond the scope of this paper.

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APPENDIX
THE PRESSURE FIELD NEAR AN OSCILLATING BUBBLE

A much-cited paper in the literature on the radiation from air guns is Keller and Kolodner (1956). They derived the expressions for the particle-velocity function and the pressure field around an oscillating bubble, using linear elasticity theory. Their expression for the pressure field contains two terms: a $1/r$ term and a $1/r^4$ term. Throughout this paper we have used an expression for the pressure field which contains only the first term. The purpose of this appendix is to show that although this second "afterflow" term does exist, it is absolutely negligible, has no place in the linear elasticity theory, and therefore had no business to be included by Keller and Kolodner (1956).

First we show that this afterflow term is not consistent with the linear elasticity theory and Newton's second law. Second, we show how Keller and Kolodner were able to make the mistake of including it in their analysis.

Keller and Kolodner began with the wave equation

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}, \quad (A-1)$$

in which ϕ is velocity-potential, and c is the (constant) speed of sound in water. This equation is valid for a homogeneous linearly compressible fluid with a bulk modulus K and density ρ, such that

$$c^2 = \frac{K}{\rho}. \quad (A-2)$$

Applying this equation to the radiation field around a spherical oscillating bubble, and assuming that there are only outgoing waves, they gave the solution

$$\phi = \frac{1}{r} \cdot f(t_1), \quad (A-3)$$

where

$$t_1 = t - \frac{r - a_0}{c}, \tag{A-4}$$

a_0 is the equilibrium radius of the bubble, r is the distance from the center of the bubble, and f is an unknown function.

The particle velocity is extracted from the velocity potential as follows:

$$u(t_1) = \frac{\partial \phi}{\partial r} = -\frac{1}{r^2} \cdot f(t_1) - \frac{1}{rc} f'(t_1), \tag{A-5}$$

where the prime indicates differentiation. (Strictly speaking, particle velocity is minus the partial derivative of velocity potential with respect to distance. But we used this sign convention to maintain it the same as Keller and Kolodner's.) At this point Keller and Kolodner used Bernoulli's equation to relate the pressure field to the particle velocity. However, we can find the pressure directly using Newton's second law. First we find the particle acceleration by differentiation of equation (A–5) with respect to time

$$a(t_1) = \frac{\partial u(t_1)}{\partial t} = -\frac{1}{r^2} f'(t_1) - \frac{1}{rc} f''(t_1). \tag{A-6}$$

Next we find pressure from particle acceleration using Newton's second law:

$$p(t_1) = \int -\rho a(t_1)\, dr = -\frac{\rho}{r} f'(t_1) + \text{a constant}. \tag{A-7}$$

As r tends to infinity, $p(t_1)$ tends to zero; therefore, the constant of integration in equation (A–7) is zero, and we have

$$p(t_1) = -\frac{\rho}{r} f'(t_1), \tag{A-8}$$

which is the formula we have used throughout. There is no $1/r^4$ term.

Let us now follow the analysis used by Keller and Kolodner, continuing where we broke off [after equation (A–5)]. They related particle velocity to pressure via Bernoulli's equation:

$$\int \frac{\partial p(\rho)}{\rho} = -\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2\right], \tag{A-9}$$

which they write as

$$p(r, t) = -\rho\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2\right]. \tag{A-10}$$

The problem is that the wave equation and Bernoulli's equation are not compatible. The wave equation, with constant speed of sound, is valid for *linear* elastic fluids in which the particle velocity is very small. The Bernoulli equation does not depend upon infinitesimal deformations, and it is applicable in a much wider sense than the linear wave equation. Thus there must be a problem in reconciling these two equations. The one thing they have in common is that they have both been derived using Newton's second law.

The first step that Keller and Kolodner took in this process of reconciliation was to make the approximation

$$\frac{p(r, t)}{\rho} = \int \frac{\partial p(\rho)}{\rho}, \tag{A-11}$$

implied by their version of Bernoulli's equation. Equation (A–11) is valid only for *incompressible* fluids, in which the speed of sound must be infinite. It is approximately valid for the linear elastic case. This approximation was never mentioned in their paper and is the source of all future difficulty. When they substituted for particle-velocity *squared* into equation (A–10), they derived the following expression for pressure:

$$\frac{p(r, t)}{\rho} = -\frac{1}{r} f'(t_1) - \frac{1}{r^4} f^2(t_1) - \frac{1}{2c}\left[\frac{f'^2(t_1)}{cr^2} + \frac{2f(t_1)f'(t_2)}{r^3}\right], \tag{A-12}$$

from which they chose to retain the first two terms and drop the last two. They do not examine the relative magnitudes of the terms.

In summary, we may make the following conclusion. Since the wave equation and Bernoulli's equation are both based on Newton's second law, the solutions for the pressure wave and particle velocity function must be derivable from each other using Newton's second law, if the solutions are to be consistent. This must be true whatever level of approximation is used. If the linear elasticity theory is used, infinitesimal deformations are implied. That is, the particle velocity is very small compared with the speed of sound. It follows that particle velocity *squared* is absolutely negligible, and in the linear elasticity theory there is no room for the afterflow term.

Let us look at the spectra of the radiated pressure and particle-velocity waves, for the linear elastic case. Define the Fourier transform $F(\nu)$ of $f(t_1)$ as

$$f(t_1) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i \nu t_1} d\nu. \tag{A-13}$$

After differentiation with respect to time, we have

$$f'(t_1) = \int_{-\infty}^{\infty} 2\pi i \nu F(\nu) e^{2\pi i \nu t_1} d\nu. \tag{A-14}$$

We may take the Fourier transform of equation (A–5) to yield the spectrum of the particle-velocity function

$$U(\nu) = -\frac{1}{r}\left[\frac{1}{r} + \frac{2\pi i \nu}{c}\right] F(\nu). \tag{A-15}$$

Since there are two terms in brackets which are 90 degrees out of phase, one of which is distance-dependent while the other is not, it follows that the phase spectrum of $u(t_1)$ varies with distance r. On the other hand, if we take the Fourier transform of equation (A–8) to obtain the spectrum of the radiated pressure function, we find

$$P(\nu) = -\frac{\rho}{r} 2\pi i \nu F(\nu), \tag{A-16}$$

whose phase is invariant with distance r.